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A3_1 Distortion: The Maths Behind The Magic

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Abstract

In this paper, we have derived a mathematical model for clipped distortion, an effect commonly used to modify the sound of an electric guitar. We have then applied this model to a basic speaker circuit to produce plots of the charge and current flowing through this circuit, which allowed us to see their response to the distorted signal. From the plots, it can be seen that distortion has a significant effect on their waveform.

Introduction

Distortion is an effect commonly used to alter the sound of electric guitars and other instruments. The term itself, however, is an umbrella term to describe several different more specific distortion types. The type which will be discussed in this paper is clipped distortion; the amplitude of the input signal is more than the electrical components of the amplifier can handle, and so the signal gets clipped. This results in the odd harmonics of the signal being excited, instead of the single frequency produced by a clean tone [1].

To convert the signal to an audible form, a speaker is used. The alternating current from the amplifier passes through a linear motor which moves an artificial diaphragm back and forth at the same frequency as the alternating current. The movement of the diaphragm produces sound waves that our ears can hear [2].

Forming the Model

In order to create a model for clipped distortion, it is first necessary to understand the mathematical parameters, in addition to the underlying physics. We assume that, before the sound is modulated, the signal received from the coil in one of the guitar's pickups can be represented as a simple sine-wave:

$$V(t) = V_m \sin(\omega t + \phi), \tag{1}$$

where V is the potential difference, V_m is the amplitude, ω is the frequency, t is the time, and ϕ is some phase angle.

From the definition of clipped distortion stated in the introduction, we can write our distorted function as:

$$V(t) = V_m \sum_{n=1}^{\infty} A \sin(n\omega t), \tag{2}$$

$$A = \left(\frac{1-(-1)^n}{2}\right)\left(2^{-\left(\frac{n-1}{2}\right)}\right). \tag{3}$$

The constant A contains the physical parameters of clipped distortion: Only odd harmonics are excited and the amplitude of the harmonics decreases for increasing n (we choose by a factor of 2, but this can be altered at the author's behest).

Adding a Speaker

To simulate the response to this signal of a basic speaker circuit, we can assume that the speaker's coil is an inductor of inductance L with some resistance R (a more complex and realistic circuit could be chosen, however the underlying mathematics proves to be too complex for the paper). Thus, using Kirchoff's voltage law, we can write:

$$L\ddot{q} + R\dot{q} = V(t) \tag{4}$$

Following the standard method for solving non-homogenous differential equations, we will need to find a complimentary function and particular integral to determine our general solution. By setting $V(t) = 0$ and choosing a suitable trial function, we find that

$$q(t)_{CF} = C + De^{(-\frac{R}{L}t)}, \tag{5}$$

where C and D are constants to be found. This is the transient response: how the speaker would respond when the driving voltage is taken away.

To find the particular integral, we choose another trial function:

$$q(t)_{PI} = F\cos(\omega t) + G\sin(\omega t). \tag{6}$$

Where F and G are another set of constants. This is the steady-state response. Note that the summation has been dropped. This is to simplify the maths, it will be returned later. By running this trial function back through the differential equation, we find that the constants F and G are as follows:

$$F = -\frac{R}{(L^2n^3\omega^3 + nR^2\omega)}, G = \frac{L}{R}n\omega F. \tag{7}$$

By imposing initial conditions of $q(0) = q_0$ and $\dot{q}(0) = I_0$, we can determine the constants C and D:

$$C = q_0 + \frac{L}{R}I_0 - V_m\sum_{n=1}^{\infty}\left(AF\left(\left(\frac{L}{R}n\omega\right)^2 + 1\right)\right)$$

$$D = \omega\frac{L}{R}V_m\left(\sum_{n=1}^{\infty}(AGn) - I_0\right) \tag{8}$$

Finally, we can write the general solution:

$$q(t)_{GS} = \left(C + De^{(-\frac{R}{L}t)}\right) + \dots$$
$$+V_m\sum_{n=1}^{\infty}\left(F\cos(n\omega t) + G\sin(n\omega t)\right). \tag{9}$$

Figure (1) shows the plot of the charge and current flowing through the circuit (The current is found by taking the time-based derivative of q). For both plots, the transient response has been plotted in black and the steady-state in blue.

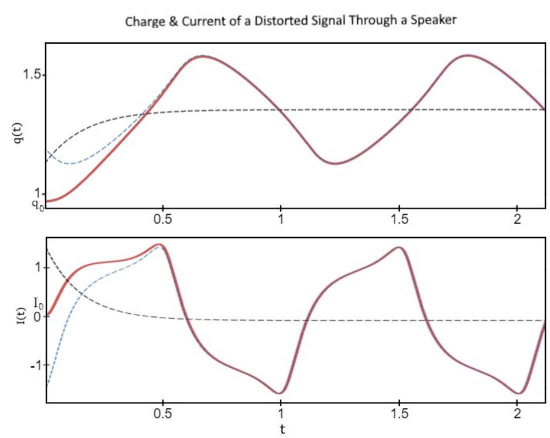

Figure 1: Plots of the response of the charge and current in a speaker circuit. Arbitrary values were chosen to aid visualization.

Conclusion

From the plots in Figure (1), we can determine the following: as expected, the transient response of the charge increases to an asymptote, whilst the current decreases. However, due to the nature of the distorted driving voltage, their waveforms have significantly deviated from the expected sinusoidal form of a clean signal.

References

[1] `https://blackstoneappliances.com/dist101.html` [Accessed 10 Oct.2022]

[2] `https://www.soundguys.com/how-speakers-work-29860/` [Accessed 21 Oct. 2022]