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P2_2 Big Ben 2: Enormous Benjamin

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Abstract

The time-keeping ability of an analogue clock in the modern age relies heavily on the functioning of the motor. In this paper we ask: In the absence of a motor as the limiter, assuming some kind of ideal time keeping implement with the power to provide an arbitrarily large amount of torque, what would be the limits of the clock's time-keeping abilities? We find the optimal material for such a construction to be Beryllium and with calculations using the shear wave speed in this material we find a 4.4km length clock hand to be a reasonable upper limit to this clock's abilities.

Introduction

On the usual scale, clocks find their bottleneck or limiter to generally be based on the motor of the clock; it is not often that a pocket watch owner considers the time it takes for the tick of the motor to travel to the end of a clock hand. This is because on the scale of a pocket watch this time may as well be negligible. However, given an ideal motor one could theoretically extend the hands of a clock to a point at which this time is no longer negligible. In this paper we consider the interesting phenomena that arise from re-scaling a clock to its extremes given a motor that provides arbitrarily large torque and from this case study hope to find a deeper understanding of the behaviour of matter waves at these scales.

With this torque, we can assume that no matter the mass of the clock hands the motor will always turn the hand. This allows us to focus on the phenomena of transverse matter waves within the material and the limit of the clock beyond the motor.

Theory

This paper's key premise is that changes from the central point of the hand to the outer edge do not happen instantaneously, the force of the motor acting upon the hand travels through the hand at a certain speed. What this means is that although our motor may keep perfect time, there is a length at which the outer edge of the clock hand will be out of sync with the tick of the motor. The turning of the hand travels up the hand as a transverse wave, travelling at a speed controlled by the properties of the material being travelled in.

The properties we are concerned with for the construction of our clock are the density ρ and rigidity (shear modulus) G of the material. We use the shear modulus rather than Young's modulus or the bulk modulus as the type of wave generated by a movement of this type is transverse. The turn of the motor generates a shear force upon the hand, acting clockwise from the point of contact with the motor.

The wave speed in a material is given by:

$$v_s = \sqrt{\frac{G}{\rho}} \tag{1}$$

We can see that for our clock we are aiming for a material with the highest rigidity to density ratio in order to have our tick travel to the end of the hand as quickly as possible. It is important to remember that the density of a material is directly influenced by the temperature of the material. So, for our highest limit clock we can set the temperature as required to find minimum density. Although exact cases for density differ between materials, it is generally the case that higher temperatures give rise to lower densities. For the purposes of this paper, we will be considering materials of temperature 20°C under 1 atm of pressure. This is the standard operating temperature of a clock making it a reasonable assumption.

Reference [1] provides a table of materials and their shear wave speed velocities. From this table we can see that Beryllium has the highest transverse wave velocity with v_s = 8880m/s. With this velocity in hand we can consider the degree of inaccuracy we are willing to allow in our clock. A clock which points to the correct second could be up to 0.5 seconds out of time. Therefore we can allow a 0.5 second travel time t for the tick of our motor to reach the end of our clock hand:

$$x = v_s t = 4.4\text{km} \tag{2}$$

Where x is the length of the hand. Substituting in our values from [1] and the 0.5 second travel time we find that the hand has a maximum length of 4.4km

Discussion

A clock with hand length 4.4km, composed of Beryllium would be a feasible timekeeping tool (provided the ideal motor). Assuming continuous motion of the clock hands in one tick of the clock (1 second) the outer edge of the clock hand will move 1/60 the circumference of the circle its radii forms. Given this we can calculate the velocity of the outer edge of our clock hand:

$$v_c = \frac{2\pi x}{60} = 460\text{m/s} \tag{3}$$

This velocity is greater than the speed of sound in air. This amount of kinetic energy would undoubtedly lead to heating in the surrounding air. The volumetric thermal expansion coefficient for Beryllium (α_v =$1.1\times10^{-6}\text{K}^{-1}$) [1] applied to (Eqn. 4) allows us to see that for every 1°K increase in temperature (T) of 1m^3 of Beryllium the volume (V) increases by $1.1\times10^{-6}\text{m}^3$.

$$\Delta V = \alpha_v V \Delta T \tag{4}$$

We can estimate that for this length of Beryllium coupled with the thickness required to ensure its structural stability, the volume would be in the hundreds of square meters. Therefore it would not be necessary for a large change in temperature for us to see a much higher shear wave velocity (Eqn. 1) due to the reduction in density. Importantly, the heating would not occur uniformly, and instead as a gradient with the fastest (outermost) sections of the hand having the highest temperature and therefore highest sheer wave speed.

Conclusion

At a modest distance of 4.4km the reaction to a force, even in a material with a particularly fast shear wave speed, would take 0.5 seconds. It is important to note that as long as the motor of the clock is set sufficiently far ahead with the shear wave delay factored in, the hand could be of any length. There are many more opportunities for research into this thought experiment; How much longer could the clock hand be considering the temperature increase caused by the high kinetic energy? How much torque is required for a hand of this size? Could more obscure or rare materials provide drastically longer clock hands?

References

[1] David R. Lide, CRC Handbook of Chemistry and Physics, 51st Edition, 1971, CRC, 14-41