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Abstract

The aim of this paper is to give scale to the idea of Graham's number in the context of binary systems and determining how much of Graham's number could theoretically be computed if it was base 2 as opposed to base 3. This will then be compared to real-world values in order to give some sense of scale to the number.

Introduction

Graham's number is related to the field of combinatorics and is the upper bound of the answer to a question in that field. There is no need to get into the reason as to why this number is important, it was picked due to the fact that it is a very large number that isn't arbitrary. The first thing to define is Knuth's up-arrow notation. This is notation that is used to describe operations greater than multiplication. One up arrow represents exponentiation[1]:

$$3 \uparrow 3 = 3^3 = 27 \tag{1}$$

A double arrow represents tetration:

$$3 \uparrow\uparrow 3 = 3 \uparrow 3 \uparrow 3 = 3^{3^3} \approx 7.625597 \times 10^{12} \tag{2}$$

A triple arrow (pentation) follows the same pattern:

$$3 \uparrow\uparrow\uparrow 3 = 3 \uparrow\uparrow 3 \uparrow\uparrow 3 = 3^{3^{\cdot^{\cdot^{\cdot^{3}}}}} = ? \tag{3}$$

The number of 3's stacked on top of one another forms a power tower $\approx 7.625597 \times 10^{12}$ high. This number is already incalculable. The pattern continues as arrows keep getting added to get to higher and higher operations. Graham's number[2] arises by starting with $3 \uparrow\uparrow\uparrow\uparrow 3$, and iterating with the given formula:

$$g_n = \begin{cases} 3 \uparrow\uparrow\uparrow\uparrow 3, & n = 1 \\ 3 \uparrow^{g_{n-1}} 3, & n \geq 2 \end{cases} [3] \tag{4}$$

This means the number of arrows in between the 3's for g_2 is the number given by g_1 which is already an incalculable number. Graham's number is given by g_{64}. This paper describes how close we could get to numbers of this magnitude if using base 2 instead of base 3.

Method

The problem must first be simplified as the size of the number represented becomes too great before even g_1 is reached, so a new base number must be chosen. The simplest notation for numbers is binary, this being base 2 numbers. And the most efficient documentation of binary numbers is digital. Each bit is double the previous number and by reading off how many bits have been written on, we can find how close to the magnitude of Graham's number we can get without losing accuracy.

Firstly, we calculate up to $2 \uparrow\uparrow\uparrow\uparrow 3$, starting with $2 \uparrow\uparrow 3$ and then moving upwards:

$$2 \uparrow\uparrow 3 = 2 \uparrow 2 \uparrow 2 = 2^{2^2} = 16 \tag{5}$$

$$2 \uparrow\uparrow\uparrow 3 = 2 \uparrow\uparrow 2 \uparrow\uparrow 2 = 2^{2^4} = 2^{16} = 65536 \tag{6}$$

$$2 \uparrow\uparrow\uparrow\uparrow 3 = 2 \uparrow\uparrow\uparrow 2 \uparrow\uparrow\uparrow 2 = 2 \uparrow\uparrow\uparrow 65536 \tag{7}$$

Equation 7 represents the fact that the fourth arrow is going to produce a tower of indices of 2's 65536 high. By calculating the exponents from the top down, it starts with:

$$2^2 \longrightarrow 2^4 \longrightarrow 2^{16} \longrightarrow 2^{65536} \longrightarrow 2^{2^{65536}} \longrightarrow ... \tag{8}$$

after completing those exponents there are 65531 left. The final step is comparing this calculation with physical values and real world examples. Numbers expressed in physics are often in base 10 meaning the base 10 numbers must be converted to base 2, to accurately compare. The conversion to check the power of 10 these values found come up with is given below, where x is the exponent of 2 and y is the exponent of 10.

$$2^x = 10^y \tag{9}$$

$$x = y \log_2(10) \tag{10}$$

The equation is used to convert the quantities described in Table 1 as powers of 2 as opposed to powers of 10, such that they can be compared to a power of 65536.

Results

Name of Quantity	Exponent of 2 (x)
Data in the World (2020)[4]	76.404
Data in the World (2025)[5]	79.73
Atoms in the World[6]	162.8 - 166.1
Atoms in the Solar System[7]	186.03
Atoms in the Galaxy[7]	222.57
Protons in the Universe[7]	265.75
Planck Volumes in the Universe[8]	614.6

Table 1: Table describing the size of real-world large quantities as an exponent of 2

Conclusion

What this shows is that even if every piece of matter was used as a bit in the theoretical digital device, the next exponent in the tower of indices discussed above is still 246.6 times greater. This means that there would need to be the number of protons in our universe to the power of 246.6, in our machine expressed as bits, just to calculate the next exponent i.e. $(\text{number of protons in the universe})^{246.6}$. Following from that the process, as shown in equation 8, is then repeated 65531 times again to get the final number. In relation to Graham's number, this calculates g_1 if the base were 2. The process of how to move to the next g value has been previously discussed, but to reiterate the magnitude, g_{64} is the final number to be calculated.

References

[1] https://bit.ly/32Lk03y [Accessed 28 November 2021]

[2] https://bit.ly/3cY35g2 [Accessed 28 November 2021]

[3] https://bit.ly/3I6WkH6 [Accessed 28 November 2021]

[4] https://bit.ly/3rpjJ0m [Accessed 28 November 2021]

[5] https://bit.ly/3CZJsyE [Accessed 28 November 2021]

[6] https://bit.ly/317zLBw [Accessed 28 November 2021]

[7] https://bit.ly/31ity5u [Accessed 28 November 2021]

[8] https://bit.ly/3I8bFa8 [Accessed 28 November 2021]