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Abstract

This paper verifies that the distribution of mass of the Earth described by the density equations outlined in the PREM model are correct by utilizing them to calculate the moment of inertia of the Earth as 8.117×10^{37} kg m^2. This was then found to be within 1% of the known value of the Earth's moment of inertia, 8.038×10^{37} kg m^2.

Introduction

The Preliminary Reference Earth Model (PREM) was developed in 1981 and is still a leading model in seismology worldwide today [1]. The one-dimensional model describes the Earth's properties, such as density, as a function of radial distance. Previously, the density functions from the PREM model were used to calculate the mass of each radial region of the Earth [2]. The calculated values are provided in Table 1 below.

Radius (km) [1]	**Mass (kg)** [2]
0.0000 - 1221.5	$M_1 = 9.843 \times 10^{22}$
1221.5 - 3480.0	$M_2 = 1.841 \times 10^{24}$
3480.0 - 5701.0	$M_3 = 2.940 \times 10^{24}$
5701.0 - 5771.0	$M_4 = 1.153 \times 10^{23}$
5771.0 - 5971.0	$M_5 = 3.334 \times 10^{23}$
5971.0 - 6151.0	$M_6 = 2.899 \times 10^{23}$
6151.0 - 6346.6	$M_7 = 3.235 \times 10^{23}$
6346.6 - 6356.0	$M_8 = 1.382 \times 10^{22}$
6356.0 - 6368.0	$M_9 = 1.587 \times 10^{22}$
6368.0 - 6371.0	$M_{10} = 1.560 \times 10^{21}$

Table 1: Mass between each radial region, $R_0 = 0$ km, $R_1 = 1221.5$ km, $R_2 = 3480$ km, etc.

The sum of the masses above closely matched the measured value of the Earth's mass which indicates that the overall density values are correct. However, to further investigate the model, we aimed to determine that this mass was accurately distributed around the Earth's centre of mass. This was achieved by using the masses to calculate the Earth's moment of inertia. The total moment of inertia of the Earth can be calculated by summing the moment of inertia values for each radial region of the Earth. This can be done because the moments of inertia for different parts of an object can be added as long as they are rotating about the same axis [3].

Method

Moment of inertia of a sphere is given by [4]:

$$I_{Sphere} = \frac{2}{5}MR^2 = \frac{2}{5}(\rho V)R^2 \tag{1}$$

Substituting in the volume of a sphere for V:

$$I_{Sphere} = \frac{2}{5}\rho\left(\frac{4}{3}\pi R^3\right)R^2 = \frac{8}{15}\pi\rho\left(R^5\right) \tag{2}$$

For a hollow sphere/shell, the moment of inertia can be calculated by calculating the moment of inertia of the outer sphere and subtracting the inertia of the inner sphere.

$$I_{Hollow} = \frac{8}{15}\pi\rho\left(R_2^5 - R_1^5\right) \tag{3}$$

Density can be given by:

$$\rho = \frac{M}{V} \tag{4}$$

Volume for a hollow sphere is given by:

$$V = \frac{4}{3}\pi\left(R_2^3 - R_1^3\right) \tag{5}$$

Thus, substituting Eq. 5 into Eq. 4 gives:

$$\rho = \frac{3M}{4\pi\left(R_2^3 - R_1^3\right)} \tag{6}$$

Substituting Eq. 6 into Eq. 3 gives:

$$I_{Hollow} = \frac{2}{5}M\frac{\left(R_2^5 - R_1^5\right)}{\left(R_2^3 - R_1^3\right)} \tag{7}$$

Thus, the moment of inertia, I_i in its general form, for a radial region, defined between R_{i-1} to R_i, can be found in Eq. 8 below:

$$I_i = \frac{2}{5}M_i\left(\frac{R_i^5 - R_{i-1}^5}{R_i^3 - R_{i-1}^3}\right) \tag{8}$$

Results

The calculation for the moment of inertia for each radial region was calculated using Eq. 8. The calculation for the first region, I_1, is provided below.

$$\begin{aligned} I_1 &= \frac{2}{5}M_1\left(\frac{R_1^5 - R_0^5}{R_1^3 - R_0^3}\right) \\ &= \frac{2}{5}(9.843 \times 10^{22})\left(\frac{1221500^5 - 0^5}{1221500^3 - 0^3}\right) \end{aligned}$$

Therefore, $I_1 = 5.875 \times 10^{34}$ kg m^2

The above calculation was repeated for each radial region, the values of which can be seen in Table 2 below.

$I_1 = 5.875 \times 10^{34}$	$I_2 = 9.272 \times 10^{36}$
$I_3 = 4.528 \times 10^{37}$	$I_4 = 2.529 \times 10^{36}$
$I_5 = 7.665 \times 10^{36}$	$I_6 = 7.102 \times 10^{36}$
$I_7 = 8.425 \times 10^{36}$	$I_8 = 3.717 \times 10^{35}$
$I_9 = 4.282 \times 10^{35}$	$I_{10} = 4.219 \times 10^{34}$

Table 2: A table containing the moment of inertia values of each radial region of the Earth in units of kg m^2.

The moment of inertia of the Earth is found to be $I_{Earth} = 8.117 \times 10^{37}$ kg m^2 by summing the contribution from each radial region.

Discussion

The known value of the moment of the inertia of the Earth is 8.038×10^{37} kg m^2 [3]. The value calculated using the PREM model is within 1% of the known value and therefore implies the model is correct. This method assumes the Earth is perfectly spherical to simplify the calculations, however, this assumption is a source of error and partly explains the discrepancy between the calculated and known values of the moments of inertia.

References

[1] `https://lweb.cfa.harvard.edu/~lzeng/papers/PREM.pdf` Dziewonski, Adam M. and Anderson, Don L. (1981) Preliminary reference Earth model. [Accessed 12 October 2021]

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[4] P. A. Tipler, Physics For Scientists and Engineers (W.H Freeman and Company, New York, 2008), 6th Edition, Chapter 9, page 295

[5] `https://scienceworld.wolfram.com/physics/MomentofInertiaEarth.html` [Accessed 18 October 2021]