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Fractals and the Koch snowflake

19th April 2020

Abstract

This paper will involve an investigation into fractals, particularly the Koch snowflake. The history of fractals, from Benoît Mandelbrot's discovery in 1975 to the modern-day and future applications will be investigated. A proof will be shown for the Koch snowflake, whereby it is proved that a shape can have an infinite perimeter but a finite surface area. MATLAB will also be used in order to show visual representations of the Koch snowflake.

History of fractals

While fractals have only entered the public domain in the last 50 years, the idea had been introduced as far back as the 17th century. Gottfried Wilhelm Leibniz was the first mathematician to introduce the concept of self-similarity (1). However, in 1872, Karl Weierstrass was the first person to show a function appropriate to be called a fractal (2). Weierstrass provided an example of a function being continuous but not differentiable. Helge von Koch improved this definition in 1904 and called it the Koch curve (now called a Koch snowflake). In the 1930s, Paul Levy and George Canto both found additional fractal curves. These were named the Levy C curve and the Cantor sets respectively. Other notable researchers in the 19th and early to mid-20th century include Wallow Sierpinski, Henri Poincare, Pierre Fatou and Gaston Julia (1).

The main issue facing these mathematicians was that there was no way of visualizing a fractal. This changed with the introduction of computers. In 1975, Benoit Mandelbrot discovered the first set of fractals, known as the Mandelbrot set. In 1980, Mandelbrot, published a paper on the Mandelbrot set, accompanied by computer generated pictures of fractals, capturing the public interest on the topic.

There is some debate among the mathematical community as to the extent of which Mandelbrot is responsible for the work in his paper (2). Brooks and Matelski first presented a paper in 1979 to several universities that included the formula $z^2 + c$, as well as a crude computer picture of the image the formula produced. Mandelbrot happened to be working at one of these universities where Brooks and Matelski presented the paper. Although he denied attending the presentation, the same formula ($z^2 + c$) (2) was prevalent in Mandelbrot's paper, challenging the authenticity of Mandelbrot's paper. The terms z and c are both defined as complex numbers. Initially, c is given a fixed value while z is given the value 0. The process is then repeated, where each new output is substituted for z. Certain values of c produce outputs that exponentially increases towards infinity. However, some values of c produce outputs that forever vary within a certain boundary. The shapes that are produced are known as the Mandelbrot set. Brooks and Matelski published their paper publicly in 1981, the year after Mandelbrot published his.

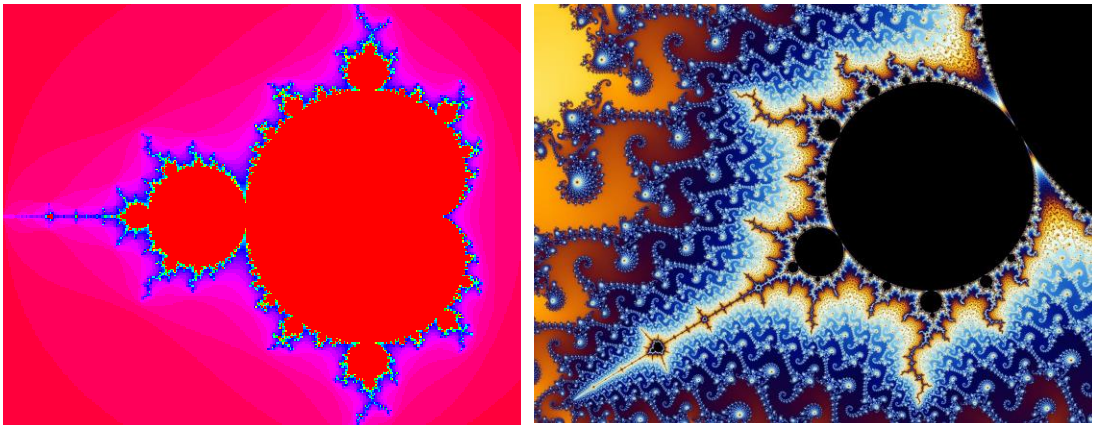

Figure 1: A couple of examples of Mandelbrot sets (3)(4)

Another point of contention is the lack of credit given to John Hubbard in Mandelbrot's paper. In 1976, Hubbard used a computer to map out Newton's method, and found another way to generate the Mandelbrot set. Hubbard was invited by Mandelbrot to discuss his research and showed him how to program a computer to plot the output of iterative functions. However, at no point in Mandelbrot's paper is Hubbard mentioned or given any credit.

There are three main ways of generating fractals (5):

- Escape time fractals – Generated by iterating a formula on each point in a given region. If the point does not diverge, it remains bounded. Examples include Mandelbrot set and the Julia set.
- Iterated function systems – These fractals are typically made up on several copies of itself, each copy then being transformed by a certain function (hence its name). The most famous example is the Sierpiński triangle.
- Random fractals – These fractals are typically generated by stochastic processes instead of deterministic processes. Examples include Brownian motion and the Brownian tree

Applications

Fractals can be found by looking at nature. While snowflakes are perhaps the obvious example; coastlines, lightning bolts and mountains are also natural examples of fractals (6). They are also increasingly being used by mathematicians in modern day technology.

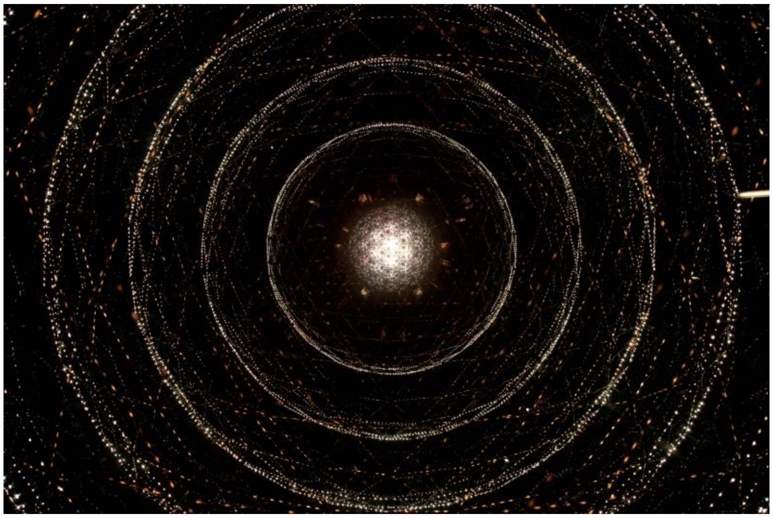

Figure 2: Galaxies have been theorized to be fractal in nature (7)

Figure 3: Lightning is a naturally occurring fractal (8)

An interesting way to explain fractals is the coastline paradox (9). Take the coastline of the United Kingdom. Depending on what scale of measurement is being used, the length of the coastline will increase. This is shown below.

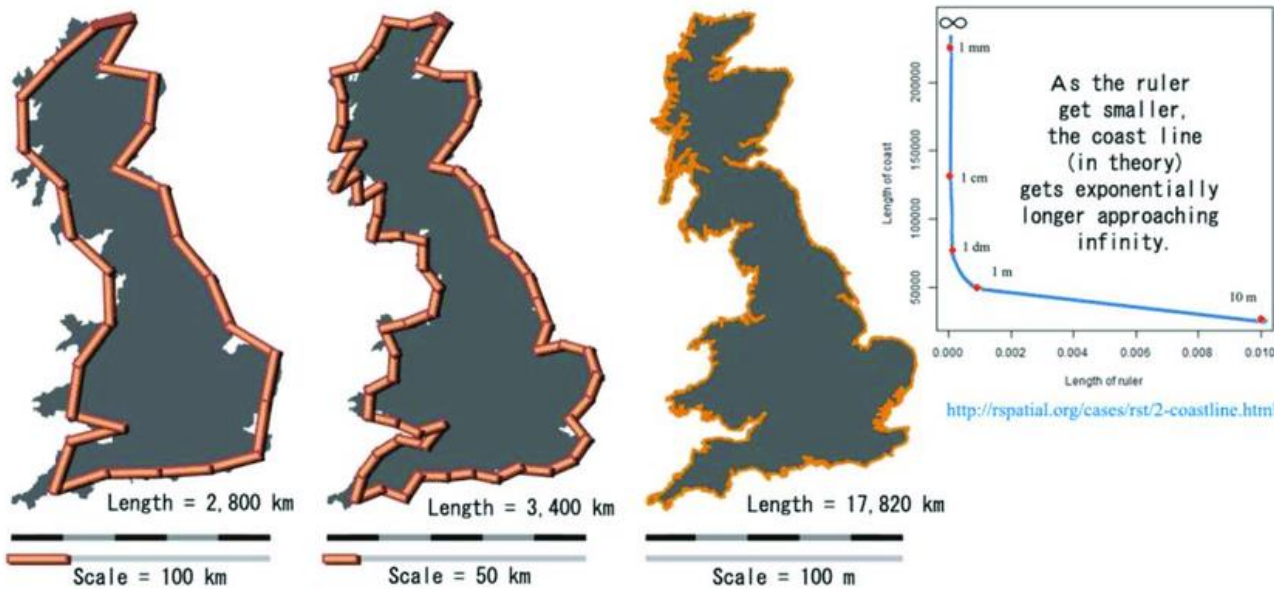

Figure 4: The coastline of the United Kingdom mapped out to three different scales (9)

When the scale decreases in size, the length of the coastline increases. As the graph above explains, as the scale reduces in size, the length of coastline gets exponentially longer, eventually approaching infinity. This means that (in theory) the United Kingdom has an infinite coastline. However, the surface area of the country can be defined. As a result, it can be said that the United Kingdom has an infinite perimeter (coastline) but a defined surface area. This is one of the most important properties of the fractal and will be proved later in the paper.

In the 1990s, fractal antennas started to be used (6). Previous antennas could only pick up the one signal they were designed for. For example, FM antennas could only pick up FM signals. Fractal antennas on the other hand can pick up many signals. This is due to the fact the shapes within a fractal antenna constantly repeat themselves at varying lengths, allowing the antenna to work for multiple frequencies rather than a single frequency. In addition, a fractal antenna takes up roughly a quarter of the space a regular antenna takes up. However, as fractals are hard to make, they tend to be expensive. In addition, while being able to pick up multiple signals, the quality of each signal tends to be less than the quality of signal from a regular antenna for its specific signal.

In biology, MIT scientist discovered that the substance chromatin is a fractal (6). As it is a fractal, it stops DNA from getting tangled with each other as it passes through. Fractals are also fundamental for the testing of a wide range of systems. These include dynamical, nonlinear, complex and chaotic systems.

A brief introduction to the Koch snowflake

One of the most famous examples of a fractal is the Koch snowflake (10). It accurately shows a fractal's properties. The number of sides for each iteration of the snowflake follows the equation $3 * 4^{n-1}$, where n is the number of iterations. To create the Koch snowflake, start with an equilateral triangle. An equilateral triangle is then added to the middle of each side of the triangle. This process is repeated at every iteration to each side of every triangle. The first 4 iterations are shown below with the number of sides for each iteration directly underneath it (11):

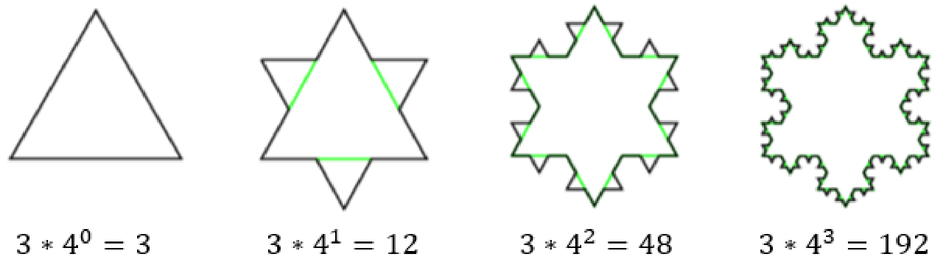

Figure 5: First four iterations of Koch snowflake (11)

As the number of sides increases, so does the perimeter of the shape. If each side has an initial length of 1 metre, the perimeter will equal 3 metres. For the second iteration, each side will have a length $\frac{1}{3}$ of a metre so the perimeter will equal $\frac{1}{3} * 12 = 4\ metres$. The length of each side for the third iteration will be $\frac{1}{9}$ of a metre so the perimeter will equal $\frac{1}{9} * 48 = \frac{48}{9} = \frac{16}{3} = 5\frac{1}{3}\ metres$.

From this, it can be concluded that for each iteration, the perimeter of the shape increases by 33%, and the total number of sides increases by a multiple of 4. This means that as the number of iterations approaches infinity, the perimeter will approach infinity.

Conversely, the surface area is finite. A simple way to grasp this concept is that if a circle is drawn encompassing the original snowflake in its first iteration, after an infinite number of iterations, the circle will still contain the snowflake. To follow on from this statement as well as the previous paragraph discussing the infinite perimeter of the Koch snowflake, mathematical proofs will be explained for both concepts below.

Koch snowflake – area is finite, and perimeter is infinite proof

Proof that the surface area S is finite:

First, a table that outlines the total area of the Koch snowflake at generation n will be provided as a base for the proof: (12)

Generation N	Area of 1 single triangle	Total number of triangles at generation N	Total Area added at generation N
0	A_0	1	A_0
1	$\left(\frac{1}{9}\right) A_0$	$3 * 4^0$	$\left(\frac{1}{9}\right) A_0 * 3 * 4^0$
2	$\left(\frac{1}{9}\right)^2 A_0$	$3 * 4^1$	$\left(\frac{1}{9}\right)^2 A_0 * 3 * 4^1$
3	$\left(\frac{1}{9}\right)^3 A_0$	$3 * 4^2$	$\left(\frac{1}{9}\right)^3 A_0 * 3 * 4^2$
⋮	⋮	⋮	⋮
n	$\left(\frac{1}{9}\right)^n A_0$	$3 * 4^{n-1}$	$\left(\frac{1}{9}\right)^n A_0 * 3 * 4^{n-1}$

Before the area proof is continued, explanations for how the formulas in the table above arose will be presented. Firstly, the area of 1 single triangle is:

$$\left(\frac{1}{9}\right)^n A_0$$

Where n is the n^{th} iteration and A_0 is the area of the 1 triangle at generation 0. The side lengths of each triangle respective to the other triangles in the same iteration each have the same side length. As well as this, every iteration simply splits each side of the triangle into equal parts. Therefore, the side length of a triangle will be a third of the side length of the triangle from the previous iteration. This implies that the area of 1 triangle at generation 1 will be one ninth of the area of 1 triangle at generation 0, hence, the area of 1 single triangle at the n^{th} iteration is:

$$\left(\frac{1}{9}\right)^n A_0$$

Next, the total number of triangles at generation N will be looked at. At generation 0 there is 1 triangle. After this generation, a single triangle will be added to the side of each triangle, bar the base side of the triangle. So, at generation 1, there will be 3 triangles. At generation 2, there will be 12 triangles. So, in ascending order of iterations, there will be 1,3,12,48... triangles. This is modelled by the formula for $n > 0$:

$$number\ of\ triangles\ at\ generation\ n = 3 * 4^{n-1}$$

The paper has explained how the area of 1 triangle at any given generation is calculated as well as how many triangles can be expected at that same generation. Now, the total area added at each iteration is simply the number of triangles at any given iteration multiplied by the respective area of each triangle. Which gives:

$$\left(\frac{1}{9}\right)^n A_0 * 3 * 4^{n-1}$$

Now that an explanation for how each formula in the table has been obtained, the paper can now continue with the proof for how the total surface area is finite.
So, the total surface area S is simply the sum of every iteration of the total area added at generation N, written below:

$$Total\ Surface\ Area = S = A_0 + \left[\left(\frac{1}{9}\right)A_0 * 3 * 4^0\right] + \left[\left(\frac{1}{9}\right)^2 A_0 * 3 * 4^1\right] + \left[\left(\frac{1}{9}\right)^3 A_0 * 3 * 4^2\right] + \cdots + \left[\left(\frac{1}{9}\right)^n A_0 * 3 * 4^{n-1}\right]$$

Now, multiply S by $\frac{4}{9}$ to get:

$$\frac{4}{9}S = \frac{4}{9}\left[A_0 + \left[\left(\frac{1}{9}\right)A_0 * 3 * 4^0\right] + \left[\left(\frac{1}{9}\right)^2 A_0 * 3 * 4^1\right] + \left[\left(\frac{1}{9}\right)^3 A_0 * 3 * 4^2\right] + \cdots + \left[\left(\frac{1}{9}\right)^n A_0 * 3 * 4^{n-1}\right]\right]$$

Once simplified, using the laws of indices, the power of each factor of 4 increases by 1 as does the power of $\frac{1}{9}$ which gives:

$$\frac{4}{9}S = \left[\frac{4}{9}A_0 + \left[\left(\frac{1}{9}\right)^2 A_0 * 3 * 4^1\right] + \left[\left(\frac{1}{9}\right)^3 A_0 * 3 * 4^2\right] + \left[\left(\frac{1}{9}\right)^4 A_0 * 3 * 4^3\right] + \cdots + \left[\left(\frac{1}{9}\right)^n A_0 * 3 * 4^{n-1}\right] + \left[\left(\frac{1}{9}\right)^{n+1} A_0 * 3 * 4^n\right]\right]$$

Now, subtract $\frac{4}{9}S$ from S in order to get rid of the infinite tail which gives:

$$S - \frac{4}{9}S = \left[A_0 + \left[\left(\frac{1}{9}\right)A_0 * 3 * 4^0\right] + \left[\left(\frac{1}{9}\right)^2 A_0 * 3 * 4^1\right] + \left[\left(\frac{1}{9}\right)^3 A_0 * 3 * 4^2\right] + \cdots + \left[\left(\frac{1}{9}\right)^n A_0 * 3 * 4^{n-1}\right]\right]$$
$$- \left[\frac{4}{9}A_0 + \left[\left(\frac{1}{9}\right)^2 A_0 * 3 * 4^1\right] + \left[\left(\frac{1}{9}\right)^3 A_0 * 3 * 4^2\right] + \left[\left(\frac{1}{9}\right)^4 A_0 * 3 * 4^3\right] + \cdots + \left[\left(\frac{1}{9}\right)^n A_0 * 3 * 4^{n-1}\right] + \left[\left(\frac{1}{9}\right)^{n+1} A_0 * 3 * 4^n\right]\right]$$

Many of these factors cancel each other out to give:

$$\frac{5}{9}S = \frac{5}{9}A_0 + \left[\left(\frac{1}{9}\right)A_0 * 3 * 4^0\right] - \left[\left(\frac{1}{9}\right)^{n+1} A_0 * 3 * 4^n\right]$$

Since $\left[\left(\frac{1}{9}\right)A_0 * 3 * 4^0\right]$ is the same as $\frac{3}{9}A_0$, this gives:

$$\frac{5}{9}S = \frac{5}{9}A_0 + \frac{3}{9}A_0 - \left[\left(\frac{1}{9}\right)^{n+1} A_0 * 3 * 4^n\right]$$

which gives:

$$\frac{5}{9}S = \frac{8}{9}A_0 - \left[\left(\frac{1}{9}\right)^{n+1} A_0 * 3 * 4^n\right]$$

Now, to get S, multiply by $\frac{9}{5}$ which gives:

$$S = \frac{72}{45}A_0 - \frac{9}{5}\left[\left(\frac{1}{9}\right)^{n+1} A_0 * 3 * 4^n\right]$$

When simplified, it gives:

$$S = \frac{8}{5}A_0 - \frac{9}{5}\left[A_0 * 3 * \frac{4^n}{9^{n+1}}\right]$$

Once again, when using the general laws of indices, this simplifies to get:

$$S = \frac{8}{5}A_0 - \left[A_0 * \frac{3}{5} * \left(\frac{4}{9}\right)^n\right]$$

As S is the area A at generation N, it gives:

$$A_n = \frac{1}{5}A_0\left[8 - 3\left(\frac{4}{9}\right)^n\right]$$

So, when $n \to \infty$, the 2nd term in the brackets can be removed because $\left[3\left(\frac{4}{9}\right)^{\infty}\right]$ would tend to 0 since the fraction would diminish when raised to a high power which means that when $n \to \infty$, we get:

$$A_\infty = \frac{8}{5}A_0$$

So, one can conclude this proof by saying that the final area of the Koch snowflake will be $\frac{8}{5}$ times bigger than the area of the parent triangle after an infinite number of generations, hence, the area is finite.

Proof that the perimeter L is infinite:

To start the proof, a line is split into three equal parts and a triangle is added on the middle segment. Each side of the triangle will have the same length as each segment of the line. The entire perimeter of the newly formed line with the triangle will therefore be $\frac{4}{3}$ times the size of the previous line without the triangle as one can see from the figure below: (12)

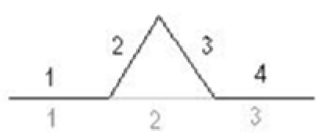

Figure 6

So, every generation will cause an increase in length of the curve of $\frac{4}{3}$ times the segment of the previous generation. Let the length of a side of the parent triangle be R. This means that the perimeter of the first triangle will be 3R. The total perimeter after each iteration will be $\frac{4}{3}$ times the size of the previous total perimeter. Let the perimeter of the triangle added at generation N be modelled by $3 * 4^{n-1} * \frac{1}{3^n} * R$. Therefore, the total perimeter L will be: (12)

$$L = 3R + \left(3 * 4^0 * \frac{1}{3^1} * R\right) + \left(3 * 4^1 * \frac{1}{3^2} * R\right) + \cdots + \left(3 * 4^{n-1} * \frac{1}{3^n} * R\right)$$

Which is the same as:

$$L = 3R + \left[\left(\frac{4}{3}\right)^0 * R\right] + \left[\left(\frac{4}{3}\right)^1 * R\right] + \left[\left(\frac{4}{3}\right)^2 * R\right] + \cdots + \left[\left(\frac{4}{3}\right)^{n-1} * R\right]$$

Now, multiply both sides by $\frac{4}{3}$ to get:

$$\frac{4}{3}L = 4R + \left[\left(\frac{4}{3}\right)^1 * R\right] + \left[\left(\frac{4}{3}\right)^2 * R\right] + \cdots + \left[\left(\frac{4}{3}\right)^{n-1} * R\right] + \left[\left(\frac{4}{3}\right)^n * R\right]$$

Now, subtract L from $\frac{4}{3}L$, this is to get rid of the infinite tail and it gives:

$$\frac{4}{3}L - L = \left[4R + \left[\left(\frac{4}{3}\right)^1 * R\right] + \left[\left(\frac{4}{3}\right)^2 * R\right] + \cdots + \left[\left(\frac{4}{3}\right)^{n-1} * R\right] + \left[\left(\frac{4}{3}\right)^n * R\right]\right]$$
$$- \left[3R + \left[\left(\frac{4}{3}\right)^0 * R\right] + \left[\left(\frac{4}{3}\right)^1 * R\right] + \left[\left(\frac{4}{3}\right)^2 * R\right] + \cdots + \left[\left(\frac{4}{3}\right)^{n-1} * R\right]\right]$$

Many of these factors cancel each other out to give:

$$\frac{1}{3}L = 4R - 3R - \left[\left(\frac{4}{3}\right)^0 * R\right] + \left[\left(\frac{4}{3}\right)^n * R\right]$$

Since $\left[\left(\frac{4}{3}\right)^0 * R\right]$ is simply just R, it gives:

$$\frac{1}{3}L = R - R + \left[\left(\frac{4}{3}\right)^n * R\right]$$

So now multiply by 3 to get:

$$L = 3\left[\left(\frac{4}{3}\right)^n * R\right] \; which\ is\ the\ same\ as: L = 3 * \frac{4}{3} * \left(\frac{4}{3}\right)^{n-1} * R$$

Which gives the final equation for the total perimeter as:

$$L = 4 * \left(\frac{4}{3}\right)^{n-1} * R$$

So, to conclude, as $n \to \infty$, L will also tend to infinity since the fraction is greater than 1 and is raised to an infinitely high power. Therefore, the final total perimeter of the Koch snowflake will be infinite after an infinite number of generations.

Visual representation of the Koch snowflake using MATLAB

MATLAB will now be used to show what the Koch snowflake looks like at its N^{th} generation. Computer software is the easiest way to see what each iteration of the Koch snowflake looks like. A computer can accurately draw out any given iteration in a matter of seconds and give a good visual representation of the fractal. The results will show what the Koch snowflake looks like for one given side of the fractal and the code used will be referenced at the end of the paper. The computer may struggle to process when $n \geq 8$ as many lines of code will have to be executed. The following MATLAB code is used: (13)

```
function []=Koch(n)
if (n==0)
   x=[0;1];
   y=[0;0];
   line(x,y,'Color','b');
   axis equal
   set(gca,'Visible','off')
else
   levelcontrol=10^n;
   L=levelcontrol/(3^n);
   l=ceil(L);
   kline(0,0,levelcontrol,0,l);
   axis equal
   set(gca,'Visible','off')
   set(gcf,'Name','Koch Curve')
end
function kline(x1,y1,x5,y5,limit)
length=sqrt((x5-x1)^2+(y5-y1)^2);
if(length>limit)
   x2=(2*x1+x5)/3;
   y2=(2*y1+y5)/3;
   x3=(x1+x5)/2-(y5-y1)/(2.0*sqrt(3.0));
   y3=(y1+y5)/2+(x5-x1)/(2.0*sqrt(3.0));
   x4=(2*x5+x1)/3;
   y4=(2*y5+y1)/3;
   % recursive calls
   kline(x1,y1,x2,y2,limit);
   kline(x2,y2,x3,y3,limit);
   kline(x3,y3,x4,y4,limit);
   kline(x4,y4,x5,y5,limit);
else
   plotline(x1,y1,x5,y5);
end
function plotline(a1,b1,a2,b2)
x=[a1;a2];
y=[b1;b2];
line(x,y);
```

Results

This code will now be executed for a variety of examples with figures displayed where necessary. When, ‘Koch(0)’, is inputted, the following is outputted:

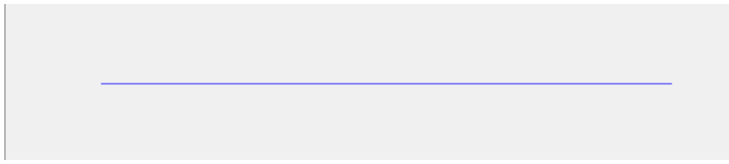

Figure 7

Looking at *figure 7*, this is the 0^{th} generation and the starting line (one side of an equilateral triangle) from which the fractal begins.
When 'Koch(1)' is inputted, the following is outputted:

Figure 8

Above, is the 1^{st} generation of the fractal. A triangle is added at equal distances from each end of the line which is the premise for each iteration to follow.
When 'Koch(2)' is inputted, the following is outputted:

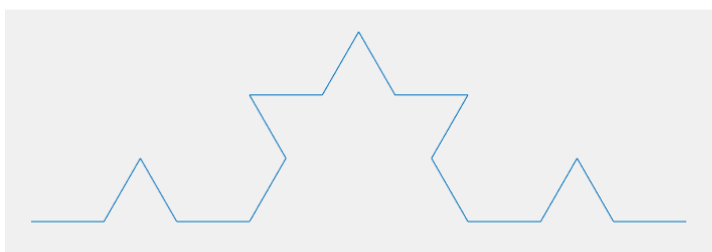

Figure 9

One can now see the fractal starting to form the 'snowflake' shape and that the perimeter is increasing. Now skipping ahead to the 6^{th} iteration, one will see the clear increase in the number of triangles in the fractal, $3 * 4^5 = 3072$, to be precise. This is a very large number of triangles and the perimeter has clearly increased dramatically. However, as proven earlier, the surface area is finite and in its 7^{th} iteration, it has the following area:

$$A_6 = \frac{1}{5}A_0\left[8 - 3\left(\frac{4}{9}\right)^6\right] = 1.595 * A_0 \; to \; 3.d.p$$

Where A_0 is the area of the first iteration.

The maximum area (as shown in earlier proof) is $1.6A_0$, so at only the 7^{th} iteration, the area is very close to its finite limit and so every iteration after this will edge ever so slightly towards its limit.
This will be the final iteration shown, $(N = 6)$ although it is important to note that there are infinitely many iterations and the better the computer, the more iterations one can look at using this MATLAB code.

When ‘Koch(6)’ is inputted, the following is outputted:

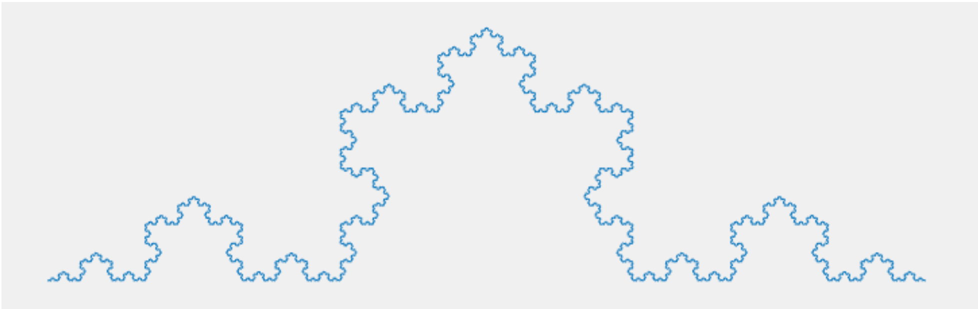

Figure 10

The following images will show the Koch snowflake for $N = 6$ slowly zoomed in to help understand how as $N \to \infty$, the pattern will recur infinitely many times.

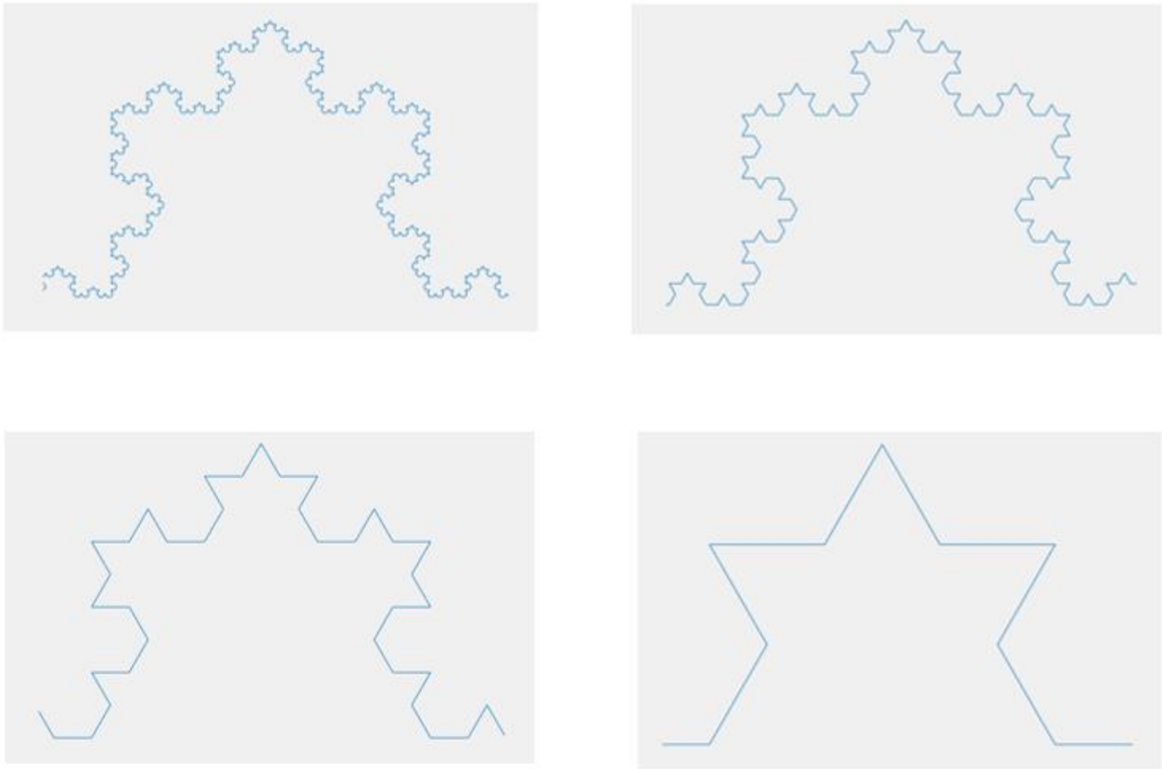

Figure 11

As is evident from *figure 10*, it is simply the same pattern repeating just getting smaller and smaller. As the N^{th} $generation$ increases, so to, will the number of triangles. The pattern will always be the same no matter how many times one zooms in.

When looking for the iteration at which the area first reaches its limit of $\frac{8}{5}A_0$, it is seen that this occurs in its 29th iteration, where the area is:

$$A_{28} = \frac{1}{5}A_0\left[8 - 3\left(\frac{4}{9}\right)^{28}\right] = 1.6 * A_0 \; to \; 3.d.p$$

This is the first iteration that the area is exactly $\frac{8}{5}A_0$ and therefore at its limit. Every iteration after this will have the same surface area. It's perimeter in the 29th iteration is:

$$L_{28} = 4 * \left(\frac{4}{3}\right)^{27} * R = 9449.436 * R \; to \; 3.d.p$$

Where R is one side length of the parent triangle. This perimeter is substantially large as the perimeter is infinite. Unfortunately, a better computer is needed to provide a visual representation of what the Koch snowflake looks like at this stage. Despite this technological limitation, one can effectively assume it would be similar to the 8th iteration but as you zoom in further the patterns of the fractal would continue for a long time. This is because the number of triangles at this stage would be:
$3 * 4^{27} = 5.404 \times 10^{16}$ triangles which is a huge number of triangles.

Modern day use of fractals, and the future

As fractals have become more understood by mathematicians and the general populace, they have become used in more and more areas of human society. Fractals are used in a wide range of technology, from detecting life via fractal analysis, to fractal designs appearing on t-shirts, computer design and neuroscience (14). There are also several areas where fractals have started to be used and their usage is likely to increase in the future. One example is in archaeology. A team of German archaeologists discovered an ancient Egyptian royal cemetery using fractals (15). As fractal geometry is naturally occurring, the scientists looked for areas of lands where the natural fractal geometry was missing, indicating human activity and landscaping, hinting at potential archaeological sites.

Another likely area that will see an increase in the use of fractal mathematics is financial trading. Currently used primarily for identifying longer term trends for stocks, more and more charting platforms now provide fractals as a trading indicator.

Furthermore, fields such as energy and biology are starting to increasingly use and understand fractals. One group of scientists are using fractal patterns of trees and leaves to enhance the amount of sunlight collected by photovoltaic cells in a solar panel. These patterns may potentially increase the efficiency of solar cells, helping the steady conversion to renewable energy.

Within biology, there are several different areas whereby the analysis has improved their understanding, have improved their understanding. In addition to the chromatin mentioned earlier, scientists have theorized bananas ripen in a fractal patter, as a result, scientists can use fractals to determine whether a banana is edible to eat or not. Another use that biologists have found is that fractal patterns in birds feathers indicate its health. Therefore, the closer a bird's feather resembles a fractal pattern, the healthier the bird is.

There are many benefits of fractals with a wide spectrum of applications. Whether fractals help to create something new or improve something already in use, there is no doubt that they change the world for the better. These applications range from helping to create a cleaner planet with improving current renewable energy technology to being used in the world of trading. Not only are fractals fascinating, they are also helping society evolve.

References

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