# An Interesting Family of Groups of Homeomorphisms of the Real Line

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**Abstract**

This paper is meant to serve as an exposition on the 2019 paper [3] by Hyde and Lodha where they managed to resolve the question posed by Rhemtulla in 1980. The authors in the paper offer a construction of families of orientation preserving homeomorphisms of the real line. Each of these families can be viewed as a finitely generated, simple, left-orderable group. Hence, each of these families satisfies the criteria that Rhemtulla laid out.

In this paper, we primarily intend to offer a simplified construction of a family of groups $G_\rho$ that the authors have constructed in the first part of their paper. The mathematics involved with the justification that these groups truly do follow the criteria laid out by Rhemtulla is not discussed in this exposition. The reader, if interested in these details, should refer to the original paper. We begin by stating the question that Rhemtulla posed in 1980 and by explaining what it means.

This is followed by a brief discussion of some preliminary concepts and tools needed for the rest of the paper. The construction is then laid out. The family of groups thus constructed contains $\aleph_0-$many groups that satisfy the criteria of Rhemtulla. The authors of the original paper use this family to construct a much bigger family of such groups where they are able to find continuum many groups that satisfy the Rhemtulla criteria. The reader, if interested should again refer to the original paper for this construction. The construction of $G_\rho$ in this paper is simplified to a great degree for the reader's ease and we hope that therefore, it would serve as a good introduction to the original paper. We conclude this paper with a historical note.

## 1 Introduction

In 1980 Rhemtulla asked the question *"Is there a finitely generated simple left orderable group?"*. The authors in their research, stumbled upon the question *"Is there a finitely generated infinite simple group of homeomorphisms of the real line?"* In the original paper, the authors refer to a paper [2] which proves that the two questions are in fact equivalent.

The original paper is fundamentally about tackling these two questions. The heart of the original paper comprises of two distinct parts. The authors in the first part of their paper provide the construction of a family of groups $G_\rho$. This is a family of $\aleph_0-$many groups that satisfy the Rhemtulla criteria (as laid out in his question). This construction was eventually used as a stepping stone, in the second part, for generating continuum many groups that satisfy the Rhemtulla criteria.

We have toiled hard to give a very simple, yet rigorous construction of $G_\rho$. The initial discussion of this construction in the original paper was difficult to understand at our undergraduate level, therefore considerable simplifications have been made. However, no compromise has been made as far as the rigour of the method is concerned. We start with $\mathrm{PL}^+([0,1])$ and Thompson's group $F$. Then, we construct $F_{\left[\frac{1}{16},\frac{15}{16}\right]}$ using $\mathcal{G}$. After this we define a quasi-periodic labelling. All of these pieces then lead to an elegant construction of $G_\rho$. As far as the second construction of continuum many groups is concerned, it is beyond the scope of this exposition to provide that construction here. If one reads the two questions posed initially in this introduction and is unable to understand them, one needs to refer to the next section for a thorough explanation. A person already familiar with the concepts may directly proceed to the construction in this exposition.

# 2 Preliminary Concepts

There are four criteria that need to be satisfied in Rhemtulla's question. These are that the object must be a group, and this group must be finitely generated, simple and left-orderable. We will begin by defining what a group is.

**Definition 2.1:** A **group** $(G, *)$ is a set $G$ together with a binary operation $*$ on $G$ satisfying the following axioms

(i) Closure: $\forall\, a, b \in G, a * b \in G$

(ii) Associativity: $(a * b) * c = a * (b * c)\ \forall\, a, b, c \in G$

(iii) Identity: $\exists\, e$ such that $e * a = a = a * e\ \forall\, a \in G$

(iv) Inverse: $\forall\, a \in G\ \exists\, b \in G : a * b = e = b * a$

In addition to this, we say the **generating set** of a group is the set of objects from which the group can be 'made'. We will also say that the **generators** are all the elements of the generating set. This leads on to the following definition of the notion of a finitely generated group.

**Definition 2.2:** A group $G$ is said to be **finitely generated** if it has a finite generating set $M = \{a_1, \ldots, a_d\}$ and consists of all products $a_{i_1}^{\epsilon_1}, \ldots, a_{i_n}^{\epsilon_n},\ i_k \in \{1, \ldots, d\}, \epsilon_k = \pm 1,\ k \in \mathbb{N}$. We write this as $G = \langle a_1, a_2, \ldots, a_d \rangle$.

This means that an object that positively answers Rhemtulla's question must be a group that is "made" by a finite number of generators. To be able to understand what a simple group is, we must be familiar with the concepts of a subgroup and a normal subgroup. After this, we can define a simple group.

**Definition 2.3:** If $H$ is a subset of a group $G$ and $H$ is also a group, then $H$ is a **subgroup** of G, denoted $H \leqslant G$.

**Definition 2.4:** A subgroup $N$ of the group $G$ is called a normal subgroup if $g^{-1}Ng = N\ \forall\, g \in G$, and we denote it by $N \trianglelefteq G$.

A **simple** group is a nontrivial group whose only normal subgroups are the trivial group and the group itself. Recall that the **trivial group** is the group containing only the identity element, often denoted as 1, $\{1\}$, or $\{0\}$.
Finally, we can say what is meant by a left-orderable group. This is the final piece in understanding the question posed by Rhemtulla.

**Definition 2.5:** A **left-ordered** group is a group $G$ together with a linear order $\leqslant$ such that if $a \leqslant b$, then $ca \leqslant cb$. A group which admits a left-ordering is called **left-orderable**.

As we have said before, in their paper, the authors provide reference to a paper that proves that the search for groups that satisfy the Rhemtulla criteria is equivalent to searching for finitely generated infinite simple group of homeomorphisms of the real line. We wish to provide the construction of such a family of homeomorphisms of the real line. The following results will be of great use for the construction later in this paper.

The following theorem is needed for the proof of theorem 2.2.

**Theorem 2.1:** *Let $G$ be a group and $H$ a finite non-empty subset of $G$. Then $H$ is a subgroup if and only if $a, b \in H \Rightarrow ab \in H$.*

A large part of this exposition relies on the properties of group homomorphisms and isomorphisms. These are defined below.

**Definition 2.6: Group homomorphisms** are maps between groups which preserve the "structure"—for groups these are the binary operations. Let $(G, \circ), (H, *)$ be groups. The map $\phi\colon G \longrightarrow H$ is called a **homomorphism** from $(G, \circ)$ to $(H, *)$, if $\forall\, a, b \in G,\ \phi(a \circ b) = \phi(a) * \phi(b)$.

An example of a group homomorphism to illustrate what is happening is given below.

**Example 2.1:** The map $\phi\colon \mathbb{R} \longrightarrow \mathbb{R}^{\times}$, defined by $\phi(x) = e^x$ is a homomorphism from $(\mathbb{R}, +)$ to $(\mathbb{R}, \times)$, since $e^{x+y} = e^x e^y$.

**Remark:** A group homomorphism is said to be a **group isomorphism**, if it is bijective, denoted $G \cong H$.

We now formally state and prove that "group isomorphisms map a subgroup of the domain to a subgroup of the codomain". This result will prove to be critical in constructing the group $G_\rho$.

**Theorem 2.2:** *Let $H_1$ and $H_2$ be subgroups of $G_1$ and $G_2$ respectively. Let $\phi\colon G_1 \longrightarrow G_2$ be a homomorphism. Then*

*(i) $\phi(H_1)$ is a subgroup of $G_2$.*

*(ii) $\phi^{-1}(H_2) = \{g_1 : \phi(g_1) \in H_2\}$ is a subgroup of $G_1$.*

*Proof.* (i) $\phi(H_1)$ is a non-empty subset of $G_2$. Let $\phi(a), \phi(b) \in \phi(H_1)$, where $a, b, \in H_1$. Then $\phi(a)\phi(b)^{-1} = \phi(a)\phi(b^{-1}) = \phi(ab^{-1}) \in \phi(H_1)$. Therefore, $H_1$ is a subgroup of $G_2$.

(ii) Since $H_2$ is non-empty, $\phi^{-1}(H_2)$ is a non-empty subset of $G_1$.

$$a, b \in \phi^{-1}(H_2) \Rightarrow \phi(a), \phi(b) \in H_2 \Rightarrow \phi(a)\phi(b)^{-1} \in H_2$$
$$\Rightarrow \phi(a)\phi(b^{-1}) \in H_2 \Rightarrow \phi(ab^{-1}) \in H_2 \Rightarrow ab^{-1} \in \phi^{-1}(H_2).$$

Therefore, $\phi^{-1}(H_2)$ is a subgroup of $G_1$.

□

Later in the construction, we will need the group $F_{\left[\frac{1}{16}, \frac{15}{16}\right]}$, which can be found using an isomorphism $\mathcal{G}$. To find $\mathcal{G}$, we will need the section formula. The group $F_{\left[\frac{1}{16}, \frac{15}{16}\right]}$ is needed for the construction of $G_\rho$.

If a point $Y = (x, y)$ lies on a line segment $\overline{XZ}$, between the points $X = (x_1, y_1)$ and $Z = (x_2, y_2)$ and satisfies the ratio $XY : YZ = m : n$, then we say that $Y$ divides $\overline{XZ}$ internally in the ratio $m : n$. The point of division has the coordinates

$$Y = \left( \frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n} \right). \tag{1}$$

We hereby direct the reader to any standard text on group theory for a more comprehensive summary.

# 3 Construction

**Definition 3.1: Homeo$^+$** $([a, b])$ is the group of all bijective, monotone continuous functions from and to the interval $[a, b]$ with continuous inverses. The group operation here, is function composition.

$\mathrm{PL}^+([a, b])$ is the set of piecewise linear, monotone increasing functions on $[a, b]$ and hence $\mathrm{PL}^+([a, b])$ is a subgroup of $\mathrm{Homeo}^+([a, b])$.

**Definition 3.2:** Given a function $g \in \mathrm{PL}^+([a, b])$, the **open support** is the set $supp(g) \coloneqq \{x \in [a, b] \mid g(x) \neq x\}$. A homeomorphism $f : [a, b] \longrightarrow [a, b]$ is said to be **compactly supported** in $(a, b)$ if $\overline{supp(f)} \subset (a, b)$, that is the image of all functions $f \in \mathrm{PL}^+([a, b])$ lie in the interval $(a, b)$.

Therefore, informally one can say that the open support of a homeomorphism is the set of all elements of the domain whose images are distinct from themselves.

**Definition 3.3:** $\mathbb{Z}[a, b, 2]$ is the set of **dyadic rationals** on an interval $[a, b]$, i.e. $\mathbb{Z}[a, b, 2] \coloneqq \left\{ a + \frac{p(b-a)}{2^q} \mid p \in \mathbb{Z},\ q \in \mathbb{N} \right\}$. A **dyadic interval** is an interval $[d_1, d_2] \subset [0, 1]$ such that $d_1, d_2 \in \mathbb{Z}[a, b, 2]$, and a **dyadic point** is a point $(x, y)$ whose coordinates $x, y \in \mathbb{Z}[a, b, 2]$.

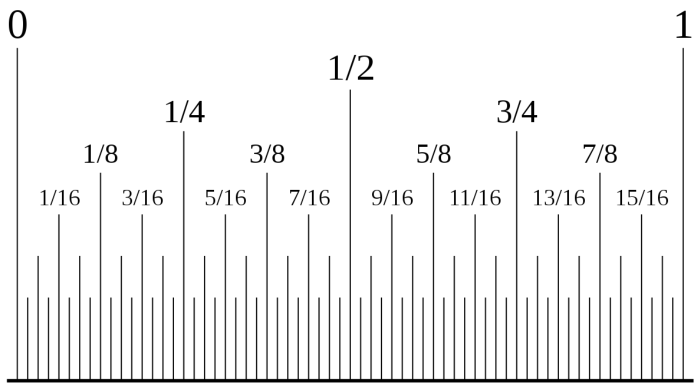


Figure 1: A visualisation of the set of dyadic rationals $\mathbb{Z}[0,1,2]$ on the interval $[0,1]$ (Credit: [5])

If $f \in \mathrm{PL}^+([a,b])$, then the points where $\frac{df(x)}{dx}$ *does not* exist are called **break points**, where $B_f = \{\text{set of all break-points of } f\}$.

**Definition 3.4: Thompson's group** is represented as $F_{[a,b]}$ on an interval $[a,b]$, and is defined as a subgroup of $\mathrm{PL}^+([a,b])$, such that

(i) $|B_f|$ is finite, and each break-point lies in the set of dyadic rationals $B_f \subset \mathbb{Z}[a,b,2]$, and

(ii) for each element $f \in F_{[a,b]}$, $\frac{df(x)}{dx}$, when existent, is a power of 2.

We will now begin with the simplest representation of Thompson's group, $F_{[0,1]}$.
Let $c_0(x) \in F_{[0,1]}$ such that $supp(c_0) = \left(0, \frac{1}{4}\right)$, $x \in \left(0, \frac{1}{4}\right) \Rightarrow c_0(x) > x$, and $c_0(x) \mid_{\left(0, \frac{1}{16}\right)} = 2x$.
$c_1(x) \coloneqq (1+x) \circ c_0(x) \circ (1-x)$. The graphs of $c_0(x)$ and $c_1(x)$ can be seen in figures 2, 3 below.

$$c_0(x) = \begin{cases} 2x, & 0 \leqslant x < \frac{1}{16} \\ x + \frac{1}{16}, & \frac{1}{16} \leqslant x < \frac{1}{8} \\ \frac{x}{2} + \frac{1}{8}, & \frac{1}{8} \leqslant x < \frac{1}{4} \\ x, & \frac{1}{4} \leqslant x \leqslant 1 \end{cases}$$

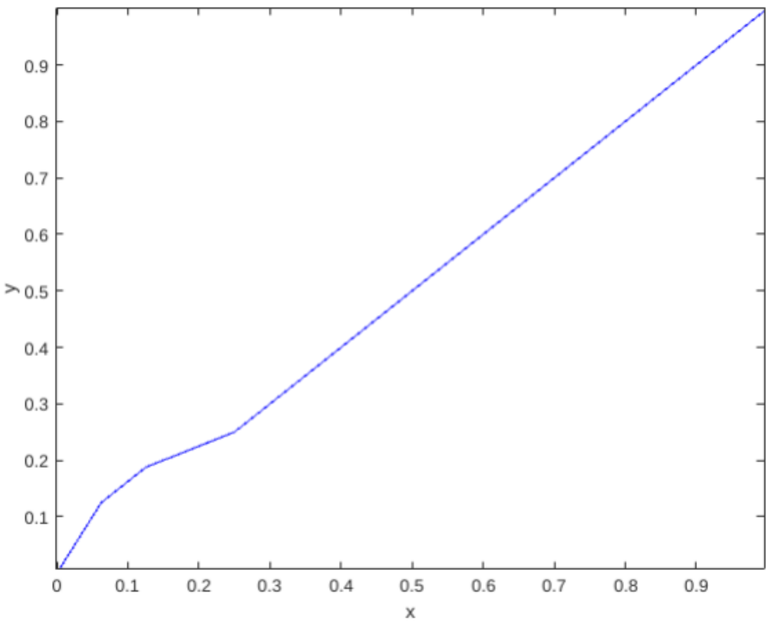


Figure 2: The graph of $c_0(x)$

$$c_1(x) = \begin{cases} x, & 0 \leqslant x < \frac{3}{4} \\ \frac{x}{2} + \frac{3}{8}, & \frac{3}{4} \leqslant x < \frac{7}{8} \\ x - \frac{1}{16}, & \frac{7}{8} \leqslant x < \frac{15}{16} \\ 2x - 1, & \frac{15}{16} \leqslant x \leqslant 1 \end{cases}$$

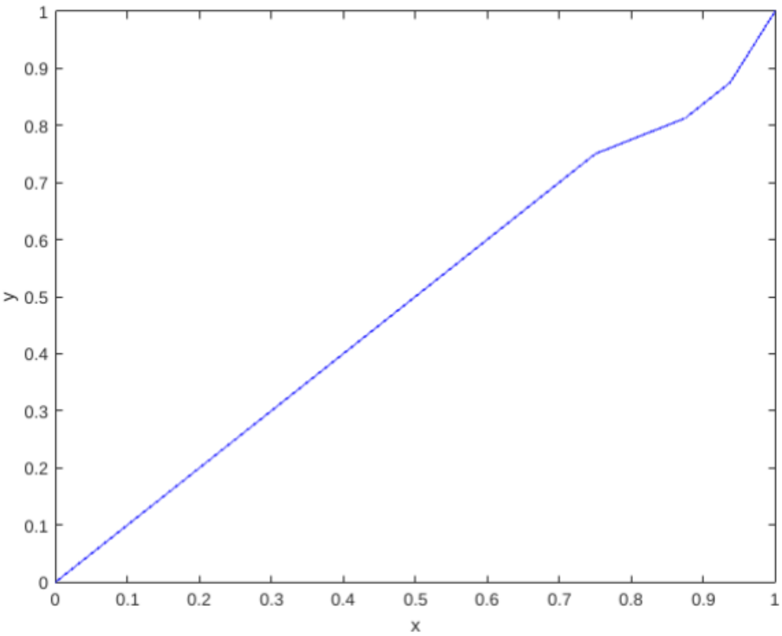


Figure 3: The graph of $c_1(x)$

We define a function $A(x) \coloneqq c_0(x) \circ c_1(x)$. This can be seen in figure 4 below.

$$A(x) = \begin{cases} 2x, & 0 \leqslant x < \frac{1}{16} \\ x + \frac{1}{16}, & \frac{1}{16} \leqslant x < \frac{1}{8} \\ \frac{x}{2} + \frac{1}{8}, & \frac{1}{8} \leqslant x < \frac{1}{4} \\ x, & \frac{1}{4} \leqslant x < \frac{3}{4} \\ \frac{x}{2} + \frac{3}{8}, & \frac{3}{4} \leqslant x < \frac{7}{8} \\ x - \frac{1}{16}, & \frac{7}{8} \leqslant x < \frac{15}{16} \\ 2x - 1, & \frac{15}{16} \leqslant x \leqslant 1 \end{cases}$$

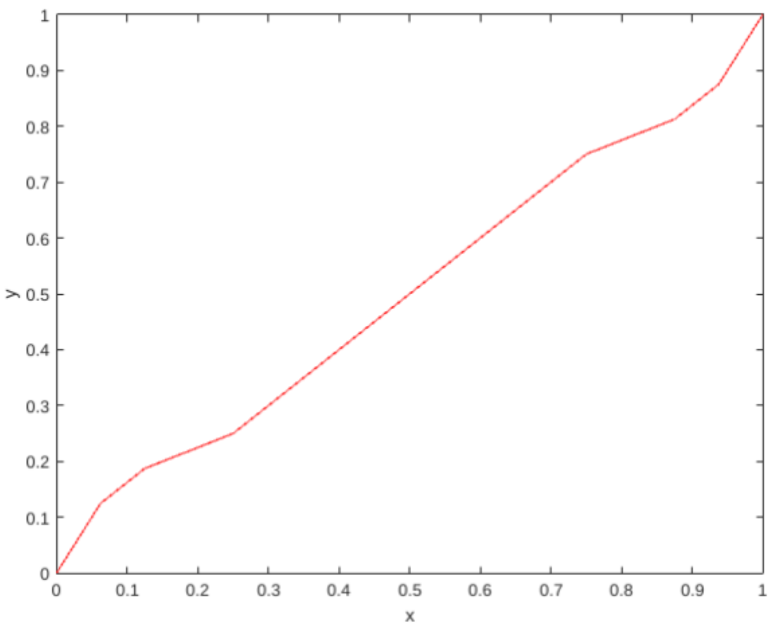


Figure 4: The graph of $A(x)$

The generators of Thompson's Group $F_{[0,1]}$ are two functions $B, C\colon [0,1] \longrightarrow [0,1]$ as follows[1]

$$B(x) = \begin{cases} \frac{x}{2}, & 0 \leqslant x < \frac{1}{2} \\ x - \frac{1}{4}, & \frac{1}{2} \leqslant x < \frac{3}{4} \\ 2x - 1, & \frac{3}{4} \leqslant x \leqslant 1 \end{cases}, \quad \text{and} \quad C(x) = \begin{cases} x, & 0 \leqslant x < \frac{1}{2} \\ \frac{x}{2} + \frac{1}{4}, & \frac{1}{2} \leqslant x < \frac{3}{4} \\ x - \frac{1}{8}, & \frac{3}{4} \leqslant x < \frac{7}{8} \\ 2x - 1, & \frac{7}{8} \leqslant x \leqslant 1 \end{cases}.$$

Therefore, we can say that $F_{[0,1]} = \langle B, C \rangle$.
For the purposes of the construction of $G_\rho$, we need the generators of the group $F_{\left[\frac{1}{16},\frac{15}{16}\right]}$. For this reason, we need a group isomorphism $\mathcal{G}\colon \mathrm{PL}^+([0,1]) \longrightarrow \mathrm{PL}^+\left(\left[\frac{1}{16}, \frac{15}{16}\right]\right)$.
We can see that if we were to map $\mathrm{PL}^+([0,1])$ on to $\mathrm{PL}^+\left(\left[\frac{1}{16}, \frac{15}{16}\right]\right)$, a certain sort of horizontal and vertical shrinking needs to take place because both the domain and image values need to lie in the interval $\left[\frac{1}{16}, \frac{15}{16}\right]$, instead of $[0,1]$.
Using intuition, one can come up with the following proportionality relations.
A point $x \in [0,1]$ should map onto a point $x' \in \left[\frac{1}{16}, \frac{15}{16}\right]$ such that $0\colon x\colon 1\colon\colon \frac{1}{16}\colon x'\colon \frac{15}{16}$, i.e. the ratio of the segment connecting $\frac{1}{16}$ and $x'$ with the segment connecting $x'$ and $\frac{15}{16}$ is $\frac{x}{1-x}$. Therefore, by applying the section formula (1), one can see that $x \longmapsto \frac{1}{16} + \frac{7}{8}x = x'$. Similarly, the point $y = f(x)$ maps to $y' = \frac{1}{16} + \frac{7}{8}y$.
Now, one can guess that the group isomorphism $\mathcal{G}\colon \mathrm{PL}^+([0,1]) \longrightarrow \mathrm{PL}^+\left(\left[\frac{1}{16}, \frac{15}{16}\right]\right)$ will be of the following form:

$$\begin{aligned} \mathcal{G}(f) &= \left(\frac{1}{16} + \frac{7}{8}x\right) \circ f(x) \circ inv\left(\frac{1}{16} + \frac{7}{8}x\right) \\ \Rightarrow \mathcal{G}(f) &= \left(\frac{1}{16} + \frac{7}{8}x\right) \circ f(x) \circ \frac{8}{7}\left(x' - \frac{1}{16}\right) =: f'(x'), \text{ and } x \longmapsto \frac{1}{16} + \frac{7}{8}x = x'. \end{aligned}$$

The proof of the isomorphicity of $\mathcal{G}$ is left to the reader as an exercise[2]. One can guess that the images of $B$ and $C$ under $\mathcal{G}$ would produce generators of $F_{\left[\frac{1}{16},\frac{15}{16}\right]}$. But, in order for this to be true, we first need to justify that all functions in the set $F_{[0,1]}$ actually do map onto functions in the set $F_{\left[\frac{1}{16},\frac{15}{16}\right]}$, and vice versa. What follows is a proof that justifies this claim.

*Proof.* We use the word dyadicity to mean the dyadic nature of the breakpoints of a function $f \in F_{[0,1]}$ or $f' \in F_{\left[\frac{1}{16},\frac{15}{16}\right]}$.
We know that $F_{[0,1]}$ is a subgroup of $\mathrm{PL}^+([0,1])$. We also know from theorem 2.2 that group isomorphisms map a subgroup of a domain to a subgroup of the codomain. Therefore, $\mathcal{G}$ will map $F_{[0,1]}$ on a subgroup of $\mathrm{PL}^+\left(\left[\frac{1}{16}, \frac{15}{16}\right]\right)$, that will be isomorphic to $F_{[0,1]}$.
Now, we show that the image $Im(F_{[0,1]}) \subseteq F_{\left[\frac{1}{16},\frac{15}{16}\right]}$ under $\mathcal{G}$ by showing the invariance of slope (of line segments between two break points) and dyadicity of break points under $\mathcal{G}$.

[1] As in [1], example 1.1, and corollary 2.6

[2] Hint: see how the functions composed on the left- and right-hand side of $f(x)$ in the definition of $\mathcal{G}$ are both linear, and hence $f$ which is piecewise linear will map onto $f'$ which will also be piecewise linear.

Let $f(x) \in F_{[0,1]}$ such that $f(x) = mx + c$ for $x$ values in between two break points.

$$\begin{aligned}
\mathcal{G}(f(x)) &= \left(\frac{1}{16} + \frac{7}{8}x\right) \circ (mx + c) \circ \frac{8}{7}\left(x' - \frac{1}{16}\right) \\
&= \frac{1}{16} + \frac{7}{8}\left(\frac{8}{7}m(x' - \tfrac{1}{16}) + c\right) \\
&= mx' + \left(\frac{1}{16} - \frac{1}{16}m + c\right)
\end{aligned}$$

Evidently, the slope is invariant. If $x$ is a dyadic point on the interval $[0, 1]$, then $x$ is of the form $\frac{p}{2^q}$. Thus, under $\mathcal{G}$, $x$ would map on to the point $x' = \frac{1}{16} + \frac{7}{8}x = \frac{1}{16} + \frac{7}{8} \cdot \frac{p}{2^q} \Rightarrow x' \in \mathbb{Z}\left[\frac{1}{16}, \frac{15}{16}, 2\right]$. Therefore, under $\mathcal{G}$, dyadicity is also preserved.
Now, we must show that $Im(F_{\left[\frac{1}{16},\frac{15}{16}\right]}) \subseteq F_{[0,1]}$ under $\mathcal{G}^{-1}$. To do this, we must find the inverse of $\mathcal{G}$, denoted $\mathcal{G}^{-1}(f')$ and conclude that for any $f' \in F_{\left[\frac{1}{16},\frac{15}{16}\right]}$ such that $f'(x') = nx' + d$ between two break points on the interval $\left[\frac{1}{16}, \frac{15}{16}\right]$, when acted on by $\mathcal{G}^{-1}$, slope and dyadicity are preserved.
One can see that:

$$\begin{aligned}
inv(\mathcal{G}(f)) &= inv\left(\frac{1}{16} + \frac{7}{8}x\right) \circ f'(x') \circ \left(\frac{1}{16} + \frac{7}{8}x\right) \\
\Rightarrow \mathcal{G}^{-1}(f') &= \frac{8}{7}\left(x' - \frac{1}{16}\right) \circ f'(x') \circ \left(\frac{1}{16} + \frac{7}{8}x\right) =: f(x), \text{ and } x' \longmapsto \frac{8}{7}\left(x' - \frac{1}{16}\right) = x.
\end{aligned}$$

Now, we need to show that slope is preserved. Consider the element $f' \in F_{\left[\frac{1}{16},\frac{15}{16}\right]}$ such that $f'(x') = nx' + d$ between two break points.

$$\begin{aligned}
\mathcal{G}^{-1}(f') &= \frac{8}{7}\left(x' - \frac{1}{16}\right) \circ (nx' + d) \circ \left(\frac{1}{16} + \frac{7}{8}x\right) \\
&= \frac{8}{7}\left(\frac{n}{16} + \frac{7n}{8}x + d - \frac{1}{16}\right) \\
&= nx + \left(\frac{n}{14} + \frac{8}{7}d - \frac{1}{14}\right)
\end{aligned}$$

It is therefore evident that the slope is invariant.
If $x'$ is a dyadic point on the interval $\left[\frac{1}{16}, \frac{15}{16}\right]$, then $x'$ is of the form $\frac{1}{16} + \frac{7p}{2^q}$. Thus, under $\mathcal{G}^{-1}$, $x'$ would map on to the point $x = \frac{8}{7}\left(x' - \frac{1}{16}\right) = \frac{8}{7}\left(\frac{1}{16} + \frac{7p}{2^q} - \frac{1}{16}\right) = \left(\frac{8p}{2^q}\right) \Rightarrow x \in \mathbb{Z}[0, 1, 2]$. Therefore, under $\mathcal{G}^{-1}$, dyadicity is also preserved.
Now, we know that Thompson's group on any interval is isomorphic to one on any other interval. Hence, the group isomorphism $\mathcal{G}$, is also a group isomorphism between $F_{[0,1]}$ and $F_{\left[\frac{1}{16},\frac{15}{16}\right]}$. Therefore we have successfully justified that the set $F_{[0,1]}$ actually does map onto the set $F_{\left[\frac{1}{16},\frac{15}{16}\right]}$ under $\mathcal{G}$, which means that, $\mathcal{G}(B)$ and $\mathcal{G}(C)$ are generators of $F_{\left[\frac{1}{16},\frac{15}{16}\right]}$. □

Now, we can use $\mathcal{G}$ to find an expression for $\nu_1 = \mathcal{G}(A(x))$, by mapping $A(x) \longmapsto \nu_1$.

$$\nu_1 = \begin{cases} 2x - \frac{1}{16}, & \frac{1}{16} \leqslant x < \frac{15}{128} \\ x + \frac{7}{128}, & \frac{15}{128} \leqslant x < \frac{11}{64} \\ \frac{x}{2} + \frac{9}{64}, & \frac{11}{64} \leqslant x < \frac{9}{32} \\ x, & \frac{9}{32} \leqslant x < \frac{23}{32} \\ \frac{x}{2} + \frac{23}{64}, & \frac{23}{32} \leqslant x < \frac{53}{64} \\ x - \frac{1}{16}, & \frac{53}{64} \leqslant x < \frac{113}{128} \\ 2x - \frac{15}{16}, & \frac{113}{128} \leqslant x \leqslant \frac{15}{16} \end{cases}$$

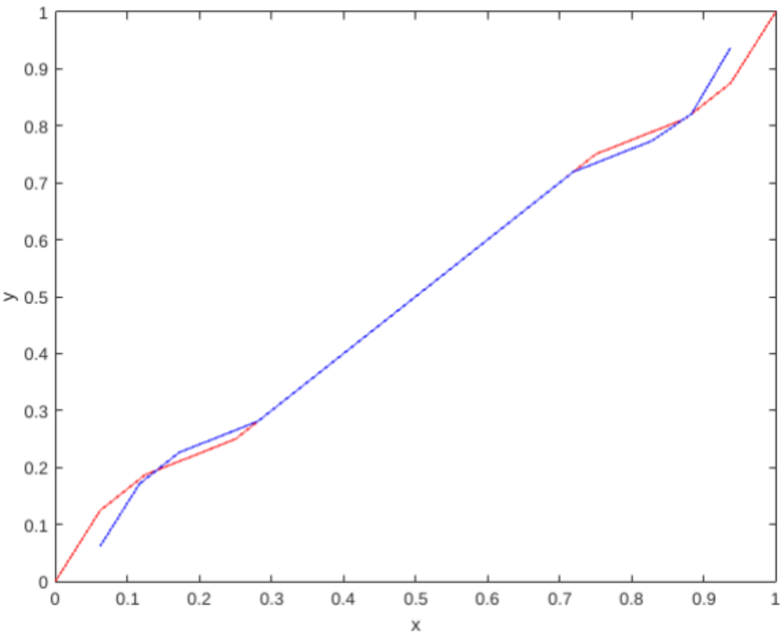


Figure 5: The graph of $A(x) \longmapsto \nu_1$

We can also now map the generators of $F_{[0,1]}$ to the generators of $F_{\left[\frac{1}{16},\frac{15}{16}\right]}$, i.e. $B(x) \longmapsto \nu_2$ and $C(x) \longmapsto \nu_3$ such that

$$\langle \nu_2, \nu_3 \rangle = F_{\left[\frac{1}{16},\frac{15}{16}\right]}, \text{ and } \mathit{supp}(\nu_2), \mathit{supp}(\nu_3) \subset \left(\frac{1}{16}, \frac{15}{16}\right).$$

The resulting expressions are as follows. The figures 6, 7 show $B(x) \longmapsto \nu_2$ and $C(x) \longmapsto \nu_3$ visually.

$$\nu_2 = \begin{cases} \frac{x}{2} + \frac{1}{32}, & \frac{1}{16} \leqslant x < \frac{1}{2} \\ x - \frac{7}{32}, & \frac{1}{2} \leqslant x < \frac{23}{32} \\ 2x - \frac{15}{16}, & \frac{23}{32} \leqslant x \leqslant \frac{15}{16} \end{cases}$$

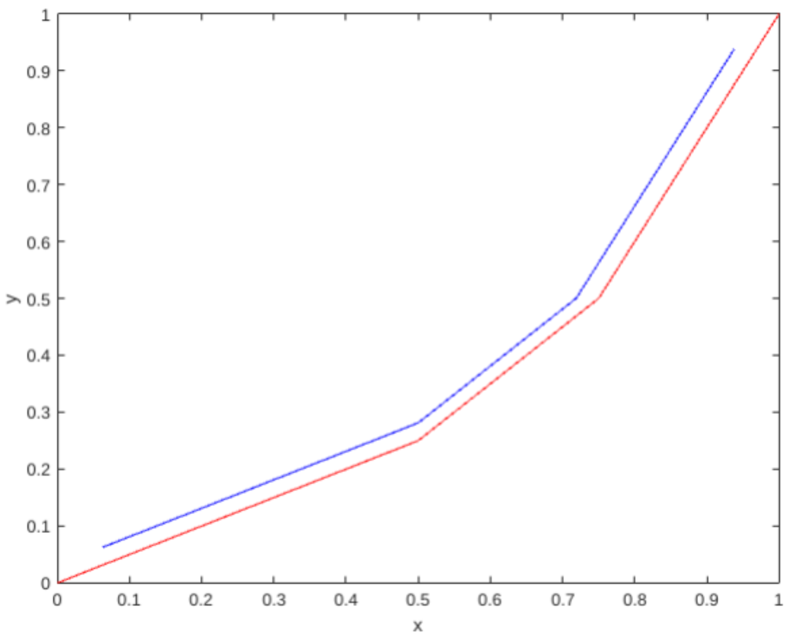


Figure 6: The graph of $B(x) \longmapsto \nu_2$

$$\nu_3 = \begin{cases} x, & \frac{1}{16} \leqslant x < \frac{1}{2} \\ \frac{x}{2} + \frac{1}{4}, & \frac{1}{2} \leqslant x < \frac{23}{32} \\ x - \frac{7}{64}, & \frac{23}{32} \leqslant x < \frac{53}{64} \\ 2x - \frac{15}{16}, & \frac{53}{64} \leqslant x \leqslant \frac{15}{16} \end{cases}$$

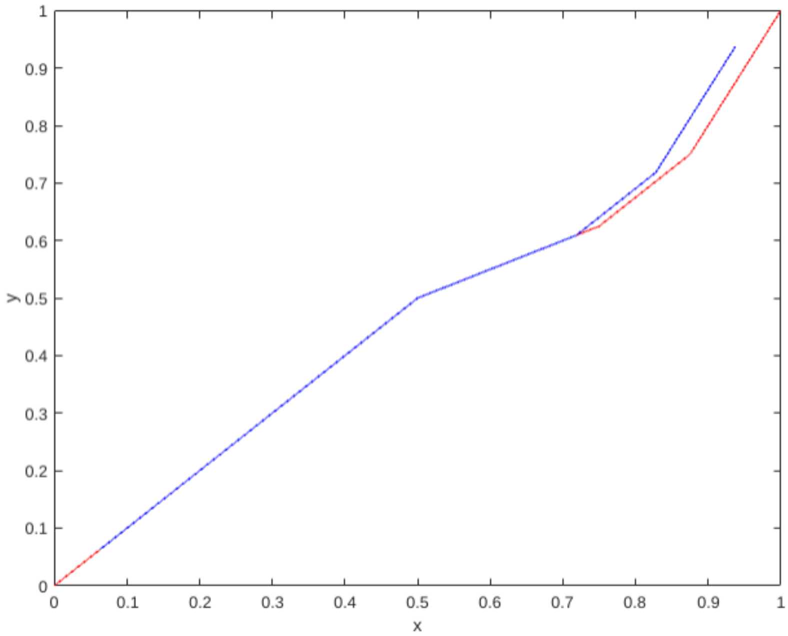


Figure 7: The graph of $C(x) \longmapsto \nu_3$

Therefore we know that $F_{\left[\frac{1}{16},\frac{15}{16}\right]} = \langle \nu_2, \nu_3 \rangle$.

We now define the functions $V_i$, $(i = 1, 2, 3)$, which are extensions of $\nu_i$ on the interval $[0, 1]$, such that the open support remains invariant. The graphs of $V_i$ can be seen in figures 8, 9, and 10, respectively.

$$V_1 = \begin{cases} x, & 0 \leqslant x < \frac{1}{16} \\ 2x - \frac{1}{16}, & \frac{1}{16} \leqslant x < \frac{15}{128} \\ x + \frac{7}{128}, & \frac{15}{128} \leqslant x < \frac{11}{64} \\ \frac{x}{2} + \frac{9}{64}, & \frac{11}{64} \leqslant x < \frac{9}{32} \\ x, & \frac{9}{32} \leqslant x < \frac{23}{32} \\ \frac{x}{2} + \frac{23}{64}, & \frac{23}{32} \leqslant x < \frac{53}{64} \\ x - \frac{1}{16}, & \frac{53}{64} \leqslant x < \frac{113}{128} \\ 2x - \frac{15}{16}, & \frac{113}{128} \leqslant x < \frac{15}{16} \\ x, & \frac{15}{16} \leqslant x \leqslant 1 \end{cases}$$

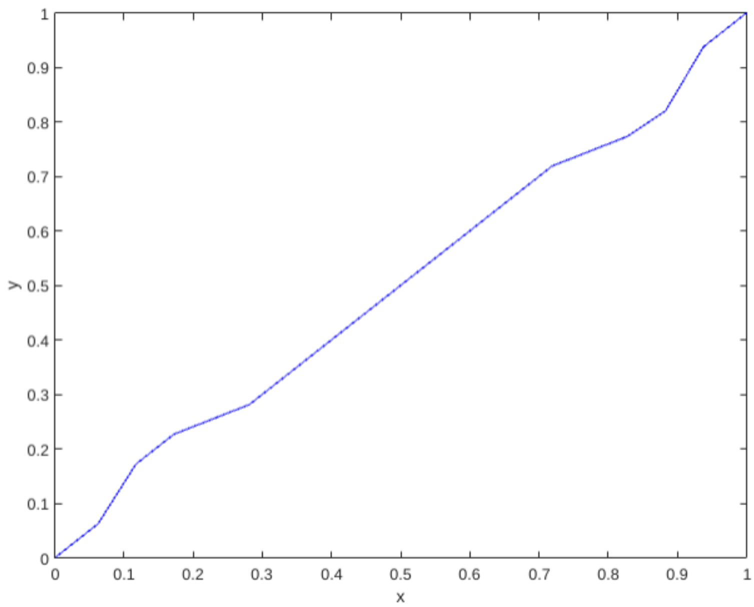


Figure 8: The graph of $V_1(x)$

$$V_2 = \begin{cases} x, & 0 \leqslant x < \frac{1}{16} \\ \frac{x}{2} + \frac{1}{32}, & \frac{1}{16} \leqslant x < \frac{1}{2} \\ x - \frac{7}{32}, & \frac{1}{2} \leqslant x < \frac{23}{32} \\ 2x - \frac{15}{16}, & \frac{23}{32} \leqslant x < \frac{15}{16} \\ x, & \frac{15}{16} \leqslant x \leqslant 1 \end{cases}$$

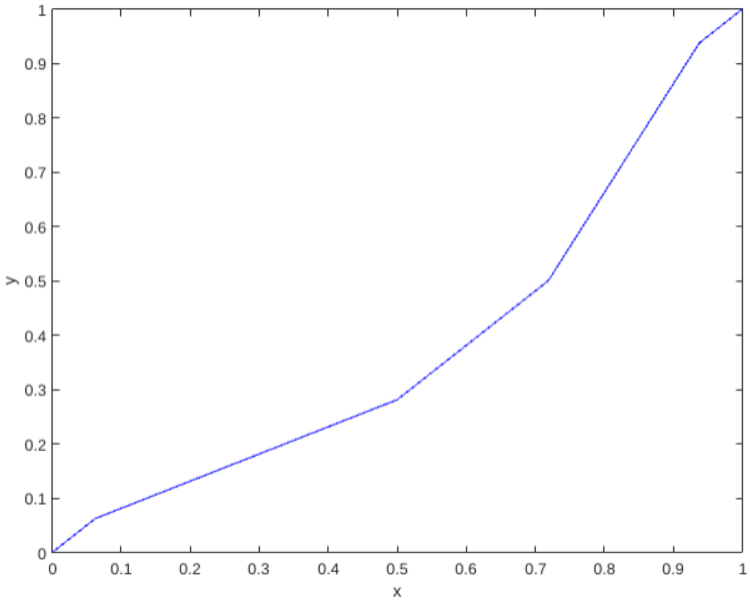


Figure 9: The graph of $V_2(x)$

$$V_3 = \begin{cases} x, & 0 \leqslant x < \frac{1}{2} \\ \frac{x}{2} + \frac{1}{4}, & \frac{1}{2} \leqslant x < \frac{23}{32} \\ x - \frac{7}{64}, & \frac{23}{32} \leqslant x < \frac{53}{64} \\ 2x - \frac{15}{16}, & \frac{53}{64} \leqslant x < \frac{15}{16} \\ x, & \frac{15}{16} \leqslant x \leqslant 1 \end{cases}$$

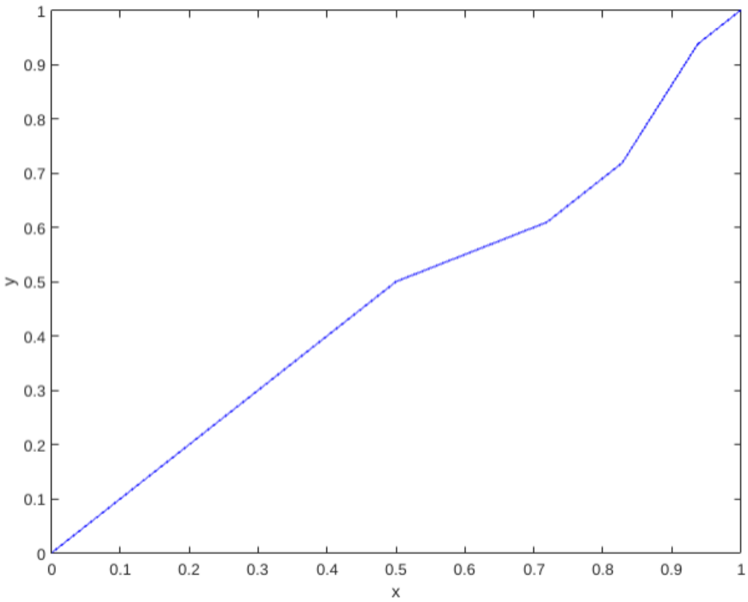


Figure 10: The graph of $V_3(x)$

The following definitions of words and blocks are needed for the definition of the final object that is needed for the construction of $G_\rho$ namely 'quasi-periodic labelling'.

**Definition 3.5:** A **labelling** is a map $\rho\colon \frac{1}{2}\mathbb{Z} \longrightarrow \{a, a^{-1}, b, b^{-1}\}$ such that

(i) $\rho(k) \in \{a, a^{-1}\},\ k \in \mathbb{Z}$, and

(ii) $\rho(k) \in \{b, b^{-1}\},\ k \in \frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}$.

A **block** refers to a set of the form $\{k, k+\frac{1}{2}, \ldots, k+\frac{n}{2}\}$. The **set of all blocks** is denoted $\mathbf{B} \coloneqq \{\{k, k+\frac{1}{2}, \ldots, k+\frac{n}{2}\};\ n \in \mathbb{N},\ k \in \mathbb{Z}\}$, and a **word** is a string of symbols defined as $W_\rho(X) = \rho(k)\rho(k+1)\ldots\rho(k+\frac{n}{2})$, where $X \in \mathbf{B}$ is a **block**.

**Definition 3.6:** A labelling is said to be **quasi-periodic** if, along with the two conditions from definition 3.5 above, the following two conditions are satisfied:

(i) $\forall\, X \in \mathbf{B},\ \exists\, n \in \mathbb{N}$ such that $\forall\, Y \in \mathbf{B}$ and $|Y| \Rightarrow W_\rho(X)$ is a **subword** of $W_\rho(Y)$ i.e. words corresponding to a block are going to repeat if the block considered gets large enough.

(ii) $\forall\, X,\ \exists\, Y \in \mathbf{B}$ such that $W_\rho(Y) = W_\rho^{-1}(X)$ i.e. there will always exist blocks whose corresponding words are inverses of words corresponding to a given block.

Now, we are faced with the conundrum of constructing such a labelling. The answer to this lies in Lemma 3.1 of [3], wherein the authors give a method of constructing them. It can also be ascertained that there exist $\aleph_0-$many such labellings.

Now, we can begin the construction of $G_\rho$. First, we define the homeomorphisms

$$\zeta_i, \chi_i \colon \mathbb{R} \longrightarrow \mathbb{R},\ i \in \{1, 2, 3\}.$$

We will construct these homeomorphisms by defining these functions on intervals of the form $[n, n+1]$. On each of such intervals, the functions would either look like $V_i$, or would look like the function we get after orientation reversal of $V_i$. Which one of these functions it would resemble on the interval $[n, n+1]$ depends on the value of quasi-periodic labelling at the mid-point of the interval, or at the beginning point, i.e. $\rho(n+1/2)$ (in the case of $\zeta$), or $\rho(n)$ (in the case of $\chi$). Thus, informally speaking, we are gluing the functions $V_i$ and their versions with reversed orientations to make the whole functions $\zeta_i$ and $\chi_i$.

Now, the question arises what exactly do we mean by orientation reversal of a function? One can convince oneself that if we have a function $f$ from and on an interval $[a, b]$, then the function $((b+a)-x) \circ f \circ ((b+a)-x)$ would be what we get if the function $f$ were reflected, first along the line $x = \frac{(b+a)}{2}$ and then reflected in the line $y = \frac{(b+a)}{2}$. Two sequential reflections of this sort are what we mean by orientation reversal.

In order for us to be able to glue functions with each other we need a way to formalise the act of translation of functions. To formalise the act of gluing functions $V_i$ (or their versions with orientations reversed) on different intervals than their actual domains and codomains, we have to translate these functions both horizontally and vertically in the Cartesian plane. In formal notation, this process would be written as the composition $(x+n) \circ V_i \circ (x-n)$. This is a translation of the functions $V_i$ (or their versions with reversed orientation). It is easy to see that both the domain and codomain change from $[0, 1]$ to $[n, n+1]$. Now, we believe it would be easy for the reader to understand the following definitions.

$$\zeta_i\,|_{[n,n+1]} = \begin{cases} (x+n) \circ V_i \circ (x-n), & \rho\left(n+\frac{1}{2}\right) = \quad b \\ (x+n) \circ \left(\frac{15}{16}+x\right) \circ V_i \circ \left(\frac{15}{16}-x\right) \circ (x-n), & \rho\left(n+\frac{1}{2}\right) = \quad b^{-1} \end{cases}$$

$$\zeta_i\,|_{[n,n+1]} = \begin{cases} (x+n) \circ V_i \circ (x-n), & \rho\left(n+\frac{1}{2}\right) = \quad b \\ \left(\frac{15}{16}+x+n\right) \circ V_i \circ \left(\frac{15}{16}-x+n\right), & \rho\left(n+\frac{1}{2}\right) = \quad b^{-1} \end{cases}$$

$$\chi_i\,|_{[n,n+1]} = \begin{cases} (x+n) \circ V_i \circ (x-n), & \rho\,(n) = \quad a \\ (x+n) \circ \left(\frac{15}{16}+x\right) \circ V_i \circ \left(\frac{15}{16}-x\right) \circ (x-n), & \rho\,(n) = \quad a^{-1} \end{cases}$$

$$\chi_i \left|_{[n,n+1]} = \begin{cases} (x+n) \circ V_i \circ (x-n), & \rho(n) = a \\ \left(\frac{15}{16} + x + n\right) \circ V_i \circ \left(\frac{15}{16} - x + n\right), & \rho(n) = a^{-1} \end{cases}\right.$$

where $n \in \mathbb{Z}$ for all the above expressions. These homeomorphisms that we have just defined form the generators of $G_\rho$. Therefore, we have successfully constructed the group $G_\rho$.

$$G_\rho \coloneqq \langle \zeta_1, \zeta_2, \zeta_3, \chi_1, \chi_2, \chi_3 \rangle$$

The fact that there exist $\aleph_0-$many quasi-periodic labellings and hence $\aleph_0-$many $G_\rho$ was discussed in Lemma 3.1 in the original paper [3]. And because of the equivalence of the two questions in our introduction, we know that $G_\rho$ satisfies the Rhemtulla criteria. Hence we are successful in providing a construction of $\aleph_0-$many groups of homeomorphisms of the real line that satisfy the Rhemtulla criteria.

# 4 Final comments and historical note

As said earlier in the paper, the authors were able to find continuum many groups that satisfy the Rhemtulla criteria. However, it is beyond the scope of this exposition to provide those constructions here. The interested reader should refer to the the original paper for further details. It is needless to say that the area of mathematics that the original paper is concerned with is a very interesting and relevant one. The following account gives a brief historical note on this field of research.

The group $F$ was first defined by Richard J. Thompson in the 1960s. It was later rediscovered by topologists Freyd and Heller, and independently by Dydak, who were researching the structure of topological spaces. Since then, $F$ has become an important object of study in geometric group theory.

The study of orderings of groups has a long history, dating back to the nineteenth century. Throughout the twentieth century many notable developments were made, with significant developments after the 1960s, following much research into linearly ordered groups. As a result, in the late 1970s to early 1980s, several detailed books and articles were written on the theory of orderable groups—most notably Kokorin and Kopytov's book, "Fully ordered groups", and Mura and Rhemtulla's book, "Orderable groups".

One consequence of the Rhemtulla criteria is that it has opened new directions of research in low dimensional topology and dynamical systems. In the past few decades a phenomenal relationship between topology and the theory of orderable groups has been discovered. Many groups of topological interest are now known to be left orderable, with examples including: torsion free abelian groups, braid groups, knot groups, fundamental groups of almost all surfaces and many manifolds in higher dimensions!

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