The Critical Bias for the Hamiltonicity Game is $\frac{(1+o(1))n}{\ln n}$

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April 2020

Abstract: This paper serves as an exposition on the theorem proved by Michael Krivelevich which states that the critical bias for the Hamiltonicity game between a Maker and Breaker can be generalised as $\frac{(1+o(1))n}{\ln n}$ [1]. Initially, basic introductions into some concepts used in Combinatorial Game Theory and Graph Theory are given. These are then built upon to create lemmas and theorems. Using these tools, the result is proved in multiple stages.

1. Introduction

Popular two-person strategy games such as Tic-Tac-Toe (Noughts and Crosses) and Hex can be generalised as having a *position* where the players take turns making *moves* to achieve a defined winning condition.

This gives way to the notion of the *positional game* used in Combinatorial Game Theory (CGT) which can be described by the following conditions:

- X - the *board*, with a finite set of elements known as *positions*
- $\mathcal{F}$ - the *winning-sets*, which are a family of subsets of X
- A criterion for winning the game

Definition 1. *A maker-breaker game is a type of positional game where two players, the Maker and the Breaker take it in turns to take unclaimed elements on a board. The objective of the Maker is to hold all the elements of a winning set, whereas the Breaker wins if they can prevent this (i.e. hold at least one element in each winning set).*

Alternatively, the *Maker-Breaker game* can be more formally defined as: a triple (H, a, b), with $H = (X, \mathcal{F})$ where H is a hypergraph.

Definition 2. *A hypergraph is a graph which is made of a set of vertices (X), the board of the game, and the set of edges (F) which join the vertices to make a winning set.*

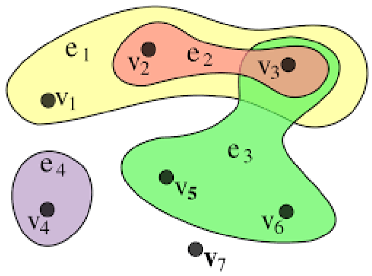

Figure 1: An example of a hypergraph with vertices $v_1, \ldots, v_7, e_1, \ldots, e_4$ and 4 edges represented by colours [2]

The parameters a and b are positive integers which represent the number of elements the Maker and Breaker can claim in each turn, respectively. They also represent the game bias.

2. Examples of Maker Breaker Games

Traditional tic-tac-toe is a strong positional game (the winner is the first player to hold all the elements of the winning set) where the second player has no chance of winning the game when playing against the perfect tic-tac-toe player. The best result for player 2 is a draw whereas player 1 can win or draw every single time. In the traditional game, both players attempt to make winning sets while also trying to prevent their opponent from making a winning set. If the game changes so that one player tries to win while the other attempts to prevent this, then the traditional game becomes a Maker-Breaker game [3].

Maker-breaker tic-tac-toe consists of two players and is played on a 3x3 grid, (the board). The aim of the maker is to pick every element of a winning set and Breaker's aim is to claim at least one element from every winning set to prevent the Maker from winning. The aim for the Maker is to pick three squares in a row and the Breaker's aim is to prevent them. The maker plays a winning strategy whereas the breaker plays a drawing strategy. Maker loses the game if Breaker can draw the game. In this game Maker has a winning strategy (i.e. they can hold all the elements of a winning set) because Maker does not need to block Breaker from obtaining a winning set. A deeper explanation of tic-tac-toe theory is explained in [4].

Hex is an example of a maker-breaker game, played between two players on a hexagonal grid. The aim is for the maker to create an unbroken chain from one side of their board - marked by their colours - to the opposite side. This is done by the players taking alternating turns to place a counter on unoccupied spaces on the board. What makes it a maker-breaker game is that there can be no draws, either Breaker creates an unbroken chain from one side to another, which will prevent Maker from creating an unbroken chain from their side to other, or Maker successfully creates the unbroken chain and they win [5].

The bias of Positional and Maker-Breaker games is, in simple terms, the number of elements placed by each player every turn. Where in the standard positional game each player would pick one element per turn, a biased game would result in a different number of elements being taken by the players on each turn.

Maker-breaker games are bias-monotone. Therefore, if Maker can win a game $(H, 1, b)$, then Maker is also able to win a game $(H, 1, b')$ where $(b' < b)$. This is because in (H, a, b), a is the number of moves Maker can make each turn and b is the number of moves Breaker can make each turn. For example, If Maker can win the game $(H, 1, 3)$, then Maker with one move per turn can win while Breaker makes three moves per turn. It follows that if Breaker has less than 3 moves each turn then Maker would still win. If Breaker's moves are reduced, then it is easy to see that Maker will also win. Therefore, the *critical bias* of the game can be defined as b^*, which is called the "*break point*". So for any game, (H, a, b^*), Maker wins for any $b < b^*$ and loses otherwise. A $(H, 1, 1)$ game has no critical bias and there is no bias for small values of n in the Hamiltonicity game.

Definition 3. *A Hamilton cycle is a graph cycle (i.e. a closed loop) through a graph that visits each vertex exactly once.*

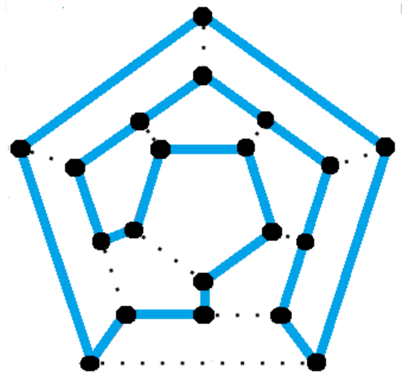

Figure 2: An example of a Hamilton cycle [6]

The *Hamiltonicity game* would be where the Maker wins if their graph contains a Hamilton Cycle in the end.

This paper looks at the Hamiltonicity game played on the edge set of a complete graph K_n.

Definition 4. *A complete graph is a graph in which every pair of vertices is connected by a unique edge. So, a complete graph K_n has n vertices and $n(n-1)/2$ edges and is only made of the vertices that are in the graph.*

3. Previous Research

The bias in maker-breaker games was first discussed in Chvatal and Erdos in 1978 [7]. They found that Maker wins the unbiased Hamiltonicity game for sufficiently large n. They also showed that Maker wins within 2n rounds. Chvatal and Erodos also proved that for $b(n) \geq \frac{(1+\epsilon)n}{\ln n}$, where $\epsilon > 0$ is an arbitrary small constant. Breaker can isolate a vertex in the $1{:}b$ game played on K_n . Later, Hefetz et al. [8] found that the minimum number of steps required for Maker to win this game was $n+2$ and finally, the optimal $n+1$ was proven by Hefetz and Stich [9].

Ballobas and Papaioannou [10] verified a conjecture by Chvatal and Erdos that there is a function $b(n)$ tending to infinity such that Maker can still produce a Hamilton cycle if they play against the bias $b(n)$. They proved that even if Breaker's bias is as large as $\frac{c\ln n}{\ln\ln n}$, Maker is able to construct a Hamilton cycle for some constant $c > 0$. Maker wins the Hamiltonicity game provided Breaker's bias is at most $\left(\frac{\ln 2}{27} - o(1)\right)\frac{n}{\ln n}$ proved by Beck [11]. Beck's result established that the $\frac{n}{\log n}$ is the order of magnitude of the critical bias in the Hamiltonicity game in view of the Chvatal-Erdos theorem about isolating a vertex. Krivelevich and Szabo [12] improved on Beck's findings and showed that the $b(n)$, the critical bias, for the Hamiltonicity game is at least $\frac{(\ln 2 - o(1))n}{\ln n}$.

This paper relies on Gebauer and Szabo's [13] findings that the critical bias for the connectivity game on K_n is asymptotically equal to $\frac{n}{\ln n}$, where Maker wins if and only if they create a spanning tree from their edges by the end of the game. It is widely believed that the critical bias for the Hamiltonicity game on K_n is asymptotically equal to $\frac{n}{\ln n}$ as well. This conjecture is described as one of the most "humiliating open problems" of the subject by Beck [4].

4. The Result

Theorem 1. *Maker has a strategy to win the* $(1\!:\!b)$ *Hamiltonicity game played on the edge set of the complete graph* K_n *on* n *vertices in at most* $14n$ *moves, for every* $b \leq \left(1 - \frac{30}{ln^{\frac{1}{4}} n}\right)\frac{n}{ln\, n}$, *for all large enough* n.

The error term expression and constants are not optimal because there is no benefit in pursuing an improved implementation of the argument. The constants and error term expression are there to ensure that the conjecture is resolved for large enough n.

Notation

For a graph $G = (V, E)$:

G - is a Graph

V - are all possible hypergraphs in G

E - are the winning sets in the graph G

Let U be a subset of V:

$N_G(U)$ - is the external neighbourhood of the subset U in graph G (i.e. the vertices outside of U that can connect to its boundary points and are still in G)

For a graph $G = (V, E)$ where V is the vertex set in G, let $U \subset V$, so $N_G(U) = \{ v \in V \backslash U : v$ has a neighbour in $U\}$.

Aside: Order notation ('big-O' and 'little-o')

Order notation is used to describe relationships between two functions or sequences which tend towards zero or infinity. For example, the notation can be used to state precisely that one function goes to zero (or infinity) faster than the other.

***Big-O notation**:* This can be used to describe how quickly a function tends to infinity.

A definition of the big-O notation proceeds as follows: If $f(x)$ and $g(x)$ are two functions which tend to zero as $x \to 0$, we say that $f(x) = O(g(x))$ ("f(x) is of order of *g(x)* as $x \to 0$"), if $|f(x)| \leq K|g(x)|$ for all $|x|$ sufficiently small where K is some positive constant.

For example:

$$2x^2 \; cos\, x + \; x^3 \; sin\, x \; = \; O(x^2)$$

While $2x^2 \; sin\, x \; + x^3 \; cos\, x \; = \; O(x^3)$ as $x \to \; 0x \to \; 0$

This is because $cos\; x \; = 1 - (x^2)/2 \; + O(x^4)$ and $sin\; x = x - (x^3)/6 + O(x^5)$

Little-o notation: This is used in order to state that one function tends to zero faster than the other.

For example:

$$e^x \; = 1 + x + \frac{x^2}{2} + \; o(x^2)$$

This means that the terms denoted as $o(x^2)$ converge to zero *faster* than x^2 as $x \to 0$

For a graph $G \; = \; (V, E)$ and a vertex subset $U \; \subset \; V$, we denoted the external neighbourhood of U in G by $N_G(U)$. For the rest of the paper it is assumed that the underlying parameter n is large enough where necessary.

Let

$$\delta_0 \; = \; \delta_0(n) \; = \frac{6}{\ln^{\frac{1}{2}} n},$$

$$\delta \; = \; \delta(n) = \; \frac{15}{\ln^{\frac{1}{4}} n},$$

$$\epsilon = \; \epsilon(n) \; = \frac{30}{\ln^{\frac{1}{4}} n},$$

$$k_0 \; = \; k_0(\, n) = \; \delta_0 n \; = \; \frac{6}{\ln^{\frac{1}{2}} n}.$$

$G \; = \; (V, E)$ is a k-expander, for a positive integer k, if $|NG(U)| \geq \; 2|U|$ for every subset $U \; \subset V$ of at most k vertices.

For a graph G we define e (a non-edge) as a *booster,* where $e = (u, v)$ of G, if adding e to G creates a graph G' which is Hamiltonian or has a maximum path longer than that of G. Boosters are added to advance a graph towards Hamiltonicity so continuously adding n boosters in a sequence clearly brings a graph to Hamiltonicity.

5. Tools

Work done by Pósa [14] was built upon to achieve the following lemma, which is used frequently in papers about Hamiltonicity and on extremal problems involving paths and cycles.

Lemma 1. *Let G be a connected non-Hamiltonian k-expander. Then at least* $\frac{(k+1)^2}{2}$ *non-edges of G are boosters.*

Proof. Lemma 8.5 of [15] or Corollary 2.10 in [16]

The next lemma shows that connected k-expander and their components are guaranteed to be relatively large size, (although k-expanders are not necessarily connected)

Lemma 2. *Let* $G = (V, E)$ *be a k-expander. Then every connected component of* G *has size at least* $3k$.

Proof (By contradiction). Let $G = (V, E)$ be a k-expander and let V_0 be the vertex set of a connected component of G which has a size less than $3k$. Now, we select an arbitrary subset $U \subseteq V_0$ with cardinality of $|U| = \min\{|V_0|, k\}$. So, the cardinality of $|U| > \frac{|V_0|}{3}$. From the definition of a k-expander, it follows that $|N_G(U)| \geq 2|U|$. We know $N_G(U) \subseteq V_0$, which implies

$$|V_0| \geq |U| + |N_G(U)| \geq 3|U|$$

However, this is a contradiction, so we have proved the lemma.

Example 1: To supplement the proof, an example of an k-expander where $k = 2$ will be used. (Note: the graph is solely to help visualisation of the proof and cannot be used as a proof itself)

As $3k = 6$, let $|V_0| = 4$

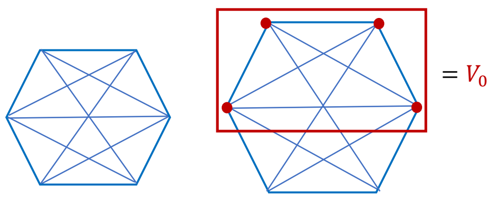

Figure 3 : A K-expander where K=2 with vertex set of connected components V_0

The arbitrary subset U will be of size 2, so we can see the cardinality of $|U| > \frac{|V_0|}{3}$

$$|U| = \min\{|V_0|, k\}$$
$$|U| = \min\{4,2\}$$
$$|U| = 2$$

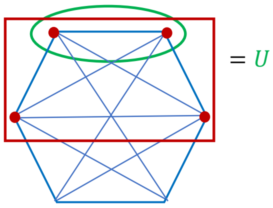

Figure 4: 2-expander with subset U of V_0

$\Rightarrow 2 > 4/3$

By definition,

$$N_G(U) \geq 2|U|$$
$$4 \geq 2|2|$$

And we know

$$N_G(U) \subseteq V_0$$

Which implies

$$|V_0| \geq |U| + |N_G(U)| \geq 3|U|$$
$$4 \geq 2 + 4 \geq 6$$

However, this is a contradiction.

Using these lemmas, the main tool of the proof can be described. This is done by applying a recent analysis of the biased minimum game by Gebauer and Szabó [13] to the game where Maker's goal is to reach a graph of **minimum** degree of at least 12.

Definition 5. *The degree of a vertex is the number of edges that connect to it.*

So, the minimum degree of the graph, $\delta(G)$, would be minimum degree of its vertices.

Maker's strategy, employed by Gebauer and Szabó, is as follows:

Maker and Breaker play a $1 : b$ game on the edges of the complete graph K_n on n vertices. For a current position of the game (with some edges of K_n being claimed by Maker and some others by Breaker), the degrees of vertex v in Maker's and Breaker's graph are denoted by $deg_{M(v)}$ and $deg_{B(v)}$ respectively.

Let us define a *component as* a connected section of Maker's graph. The component containing a vertex v can be represented by $C(v)$. This component is said to be *dangerous* if it contains at most $2b$ vertices. The *danger function, $dang(v)$*, of a vertex v with respect to the current position of the game is defined as $dang(v) := deg_{B(v)} - 2b \cdot deg_{M(v)}$.

***Strategy S**: As long as there is a vertex of degree less than 12 in Maker's graph, Maker chooses a vertex v of degree less than 12 in his graph with the largest danger value $dang(v)$ (breaking ties arbitrarily) and claims an arbitrary unclaimed edge e containing v.*

If Maker claims an edge e due to a vertex v in the strategy above, it can be said that e *is chosen by* v*.* This gives way to the following theorem:

Theorem 2. ([13], Theorem 1.2) *In a* $(1{:}\,(1-eps)n/\ln n)$ *- game played on the edge set of the complete graph* K_n*, on* n *vertices, strategy S guarantees Maker minimum degree of at least 12 in his graph.*

It is critical that Maker can reach degree at least 12 at v, for every vertex in the graph, when the substantial part of the edge at v is still unclaimed, (as stated in the following lemma).

Lemma 3. *In a* $(1{:}\,(1-eps)n/\ln n)$ *- game played on the edge set of the complete graph* K_n*, on* n *vertices, strategy S guarantees that for every vertex* $v \in [n]$ *Maker has at least 12 edges incident to* v *before Breaker accumulates at least* $(1-delta)n$ *edges at* v*.*

Proof. To prove this Lemma, a modification of the proof of Theorem 1.2 of [13] is used. This theorem is:

Let $c = c(n) < \frac{ln\, ln\, n}{3}$*. Maker has a strategy to build a graph with minimum degree at least* c *while playing against Breaker with bias* $b := \left(ln\, n - ln\, n - (2c+3)\right)\frac{n}{ln^2 n}$*, provided* n *is large enough.*

The proof of the above (discussed in [13]) analyses a game that ends when either all vertices have degree at least c in Maker's graph (i.e. Maker wins) or one vertex has degree at least $n - c$ in Breaker's graph (i.e. Breaker wins). This argument can be used for a slightly different game in which Breaker wins if they accumulate at least $(1-\delta)n$ edges at a vertex whose Maker degree is still less than 12. Then, the danger of the last vertex v_g before Break's final move can be found to be at least $(1-\delta)n - 12 - b$. Finally, one can check that the danger of the original set I_{g-1} before the game started is positive.

6. The Proof

Proving Theorem 1 is done in three stages. The first stage is where Maker creates a k_0-expander in a linear number of moves. The second stage is where Maker ensures that the graph is connected in at most $O(n/k_0)$ moves. Finally, the graph is turned into a Hamiltonian one, using at most n further moves.

Stage 1: Creating an expander

Strategy S (Gebauer-Szabo's winning strategy) for the minimum degree 12 game guarantees a minimum degree of 12 or more in Maker's graph. In addition to this, it ensures that the game is flexible enough so that Maker can pursue the goal of creating a good expander from its edges quickly.

Maker increases the degree of a vertex whose current degree in the graph is still less than 12 by one if the game is played at this stage. Therefore, Maker will win the game in at most $12n$ moves. At each round Maker is allowed to choose an edge e incident to its vertex of minimum degree v arbitrarily. Maker's freedom of choice can be specified as Maker claims a random edge e incident to v each time. This random choice allows us to prove that Maker has a strategy to create a good expander quickly.

Lemma 4. *Maker has a strategy to create a k_0-expander in at most $12n$ moves.*

Proof. Maker augments **Strategy S** by choosing a random edge incident to a vertex at each round. This new augmented strategy can be called **Strategy S'**.

Strategy S': *As long as there is a vertex of degree less than 12 in Maker's graph, Maker chooses a vertex v of degree less than 12 in his graph with the largest danger value $dang(v)$ (breaking ties arbitrarily) and claims a* ***random*** *unclaimed edge e containing v.*

We are considering a game where Maker and Breaker are playing random strategies with no chance moves. This is enough to prove that Maker's strategy in creating k-expanders with positive probability succeeds. Therefore, the game continues until the minimum degree in Maker's graph is at least 12. We also know from Lemma 4 that the duration of the game does not exceed $12n$. As a result, we can prove that Maker's strategy succeeds with probability approaching 1.

Suppose Maker's graph, M, is not a k-expander. Then there is a subset A of size $|A| = i \leq k_0$ in graph M after the end of stage 1 where $N_M(A)$, (the external neighbourhood of A in M), is contained in a set B of size at most $2i - 1$. After applying **Strategy S'**, the minimum degree of M is 12 so we can assume that $i \geq 5$ and that there are at least $6i$ edges of M incident to A.

Consider an edge e, where $e = (u, v)$. Now assume that e was chosen by $v \in A \cup B$ in the game. By Lemma 3, when choosing e Breaker's degree at v was at most $(1 - \delta)n$, while Maker's degree at v was at most 11. At that point of the game, there were at least $\delta n - 12$ unclaimed edges incident to v. Therefore, the probability that Maker chose an edge at v whose second endpoint belongs to $A \cup B$ is at most $\frac{|A \cup B| - 1}{\delta n - 12}$, these $6i$ edges incident to A will end up entirely in $A \cup B$ is at most $\left(\frac{3i-2}{\delta n - 12}\right)^{6i}$. Now we can sum over all relevant values of I and derive the probability the Maker's strategy fails to create a k-expander.

Therefore, the probability the Maker's strategy fails to create a k_0-expander is at most:

$$\sum_{5 \leq i \leq k_0} \binom{n}{i} \binom{n-i}{2i-1} \left(\frac{3i-2}{\delta n - 12}\right)^{6i} \leq \sum_{5 \leq i \leq k_0} \left[\frac{en}{i} \left(\frac{en}{2i}\right)^2 \left(\frac{4i}{\delta n}\right)^6\right]^i$$

$$= \sum_{5 \leq i \leq k_0} \left[4^5 e^3 \left(\frac{i}{n} \right)^3 \frac{1}{\delta^6} \right]^i .$$

$g(i)$ denotes the i-th term.

Then for $5 \leq\ i \leq \sqrt{n}$ we have: $g(i) \leq \left(O(1)(\ln n)^{\frac{3}{2}} n^{-\frac{3}{2}} \right)^6 =\ o\left(\frac{1}{n}\right)$.
$g(i) \leq \left(\frac{4^5 e^3 \delta_0^3}{\delta^6} \right)^{\sqrt{n}}$ is the estimate for $\sqrt{n} \leq i \leq k_0$.

These equations imply that Maker's strategy fails with negligible probability. Further, this means that Maker has a positive probability, (almost certain), of creating a k_0-expander in the first $12n$ moves.

Stage 2: Creating a connected expander
Suppose the Maker's graph is not yet connected by the end of Stage 1, it can be connected easily in a minimal number of moves. As proved earlier in Lemma 2, if M is a k_0-expander, all connected components of M are of a size of at least $3k_0$. At most, in the next $\frac{n}{3k_0} - 1$ rounds the maker can claim an arbitrary edge between two of its connected components. Looking at the complete graph, we can see there are at least $9k_0^2\ = \frac{324n^2}{\ln(n)}$ edges between any two such components. We can also observe that Breaker will have a maximum of $\left(12n + \frac{n}{3k_0}\right).b < \frac{13n^2}{\ln(n)}$ edges on the board claimed altogether. From this, we can see that Beaker will not be able to prevent Maker from claiming a full winning set. This stage lasts at most $\frac{n}{3k_0} - 1 <\ n$ rounds.

Stage 3: Completing a Hamilton cycle
Stage 1 ends with Maker creating a k_0- expander. It follows that their graph at every subsequent round receives this expansion property. After Stage 2, Maker's graph is already connected.

However, by Lemma 1 at any round of Stage 3, Maker's graph is either already Hamiltonian, or has at least $\frac{k_0^2}{2}$ boosters. Maker would then go on to continuously add boosters in the next n rounds, until Hamiltonicity is achieved.

On the other hand, Breaker does not have enough edges on the board to block all of Maker's boosters during these rounds. Therefore, the game lasts at most $12n + n + n = 14n$ rounds, during which Breaker puts at most $14n \cdot b \leq \frac{14n^2}{\ln n}$ edges. This is less than the $\frac{k_0^2}{2}$ boosters that Maker puts down.

Consequently, at any round of Stage 3, there exists an available booster for the respective Maker graph, and this would be claimed by Maker.

7. Conclusion

We employed the method of quickly creating an expander first which has the potential to be applied to other biased combinatorial games. The strategy used in the argument is random and does not provide an explicit strategy for Maker to win the Hamiltonicity game close to the critical bias.

The answer given is not the optimum however the purpose of the paper is not to find the optimum. The answer to the critical bias does not change and therefore does the optimum values are not required for the critical bias to hold.

The strategy can be generalised to firstly, finding an expander quickly by implementing Gebauer-Szabo's strategy. This strategy only fails with negligible probability. Second, Maker connects their graph. This can be done within a maximum number of rounds calculated in Stage 2. Finally, any number of required boosters can be continuously added to the graph until Hamiltonicity is achieved. From this winning strategy, the critical bias for the Hamiltonicity game between Maker and Breaker can be generalised as $\frac{(1+o(1))n}{\ln n}$ is proven true.

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