There is no Diophantine Quintuple

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Abstract

This paper will consider the works of He, Togbé and Ziegler along with Fujita to provide an expository explanation of the proof that Diophantine quintuples do not exist. The proof of this is an extension of Dujella's work on proving that there is no Diophantine sextuple and that there are at most finitely many Diophantine quintuples.

1. Introduction

A Diophantine m-tuple is a set of m positive integers $\{a_1, a_2, \dots, a_m\}$ such that $(a_i \cdot a_j) + 1$ is a perfect square for all $1 \leq i < j \leq m$.

Example 1: the set $\{1, 3, 8, 120\}$ is a quadruple since

$$1\times 3 + 1 = 2^2$$
$$1\times 8 + 1 = 3^2$$
$$3\times 8 + 1 = 5^2$$
$$1\times 120 + 1 = 11^2$$
$$3\times 120 + 1 = 19^2$$
$$8\times 120 + 1 = 31^2$$

are all perfect squares.

Diophantus was the first to explore the problem of finding four numbers such that the product of any two of them increased by unity is a perfect square. He was the first to find the rational non-integral Diophantine quadruple $\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$. This set was extended by Euler by adding a fifth rational non-integral number, $\frac{777480}{8288641}$. More recently, the first set of six positive rational numbers was found by Gibbs. However, the first Diophantine quadruple consisting of integers $\{1, 3, 8, 120\}$ was found by Fermat.

In 2004, it was proved by Dujella that there does not exist a sextuple and that there exists at most a finite number of Diophantine quintuples. This was expanded on in 2016 when He, Togbé and Ziegler showed there is no existence of a Diophantine quintuple. This expository paper will be exploring the proof of why no integer Diophantine quintuple exists.

Theorem 1. *There does not exist a Diophantine quintuple.*

When proving whether Diophantine quintuples exist or not, an element of great importance is the extensibility and existence of Diophantine m-tuples. For any fixed pair of integers $a, b > 0$ such that $ab + 1 = r^2$ is a perfect square, i.e. $\{a, b\}$ is a Diophantine pair, it was proved by Euler that a third element can always be added to $\{a, b\}$ to obtain a Diophantine triple in the form $\{a, b, a + b + 2r\}$ where $r = \sqrt{ab + 1}$.

In every Diophantine pair $\{a, b\}$ there exists infinitely many $c > 0$ where $c \in \mathbb{Z}$ such that $\{a, b, c\}$ is a Diophantine triple. Euler further noted that by adding $4r(a + r)\,(b + r)$ to the Diophantine triple, a Diophantine quadruple $\{a, b, a + b + 2r, 4r(a + r)\,(b + r)\}$ can be attained. The result of this proves the existence of an infinite number of Diophantine quadruples.

Allow there to be a Diophantine triple $\{a, b, c\}$ which does not have to be of the form $\{a, b, a + b + 2r\}$, that is $s = \sqrt{ac + 1}$ and $t = \sqrt{bc + 1}$ are both perfect squares. The mathematicians, Arkin, Hoggatt and Strauss [1] found that by adding

$$d_+ = a + b + c + 2abc + 2\sqrt{(ab + 1)(ac + 1)(bc + 1)} = a + b + c + 2abc + 2rst$$

to the Diophantine triple $\{a, b, c\}$, a Diophantine quadruple is obtained:

$$\{a, b, a + b + 2r, 4r(a + r)(b + r), a + b + c + 2abc + 2rst\,\}.$$

A Diophantine quadruple of this form is referred to as a *regular Diophantine quadruple*. Currently, all known Diophantine quadruples are of this form.

Theorem 2. *Any Diophantine quintuple contains a regular Diophantine quadruple.*

Proof. This theorem was proved rigorously in Fujita's paper [2].

Conjecture 1. *If* $\{a, b, c, d\}$ *is a Diophantine quadruple such that* $d > max\{a, b, c\}$, *then* $d = d_+$.

This conjecture was verified by He and Togbé for triples of the form $\{k, A^2k + 2A, (A + 1)^2k + 2(A + 1)\}$ with two parameters k and A where $2 \leq A \leq 10$ or $A \geq 52330$. They, as well as Fujita, showed that a triple can at most be extended to a regular quadruple. Furthermore, all known results carried out by mathematicians support this conjecture (see Table 1, page 3 in [3]).

A proof for Conjecture 1 is yet to be found as it is unknown whether there are infinitely many irregular Diophantine quadruples or not.

Lemma 1. *If* $\{a, b, c, d, e\}$ *is a Diophantine quintuple with* $a < b < c < d < e$ *then* $d = d_+$.

This was found by Fujita [2] and it is an important lemma contributing towards the proof of Theorem 1.

2. Supplementary Results

For a Diophantine triple $\{a, b, c\}$, d_+ and d_- are defined by

$$d_+ = d_+(a, b, c) = a + b + c + 2abc + 2\sqrt{(ab + 1)(ac + 1)(bc + 1)}$$
$$d_- = d_-(a, b, c) = a + b + c + 2abc - 2\sqrt{(ab + 1)(ac + 1)(bc + 1)}\ .$$

By letting $ab + 1 = r^2$, $ac + 1 = s^2$ and $bc + 1 = t^2$, we have

$$ad_\pm + 1 = (rs \pm at)^2$$
$$bd_\pm + 1 = (rt \pm bs)^2$$
$$cd_\pm + 1 = (cr \pm st)^2$$

Without loss of generality, we may assume that $a < b < c$. Then the following two lemmas will be used regularly in this paper.

Lemma 2. *If $\{a, b, c\}$ is a Diophantine triple with $a < b < c$, then*

$$c = a + b + 2r \text{ or } c > 4ab.$$

Lemma 3. $4abc + c < d_+(a, b, c) < 4abc + 4c$.

Proof. We will start with the first inequality

$$\begin{aligned} d_+ &= a + b + c + 2abc + 2\sqrt{(ab + 1)(ac + 1)(bc + 1)} \\ &> a + b + c + 2abc + 2\sqrt{(ab)(ac)(bc)} = a + b + c + 4abc \\ &> c + 4abc \end{aligned}$$

For the second inequality, we collect all non-square root terms on the left-hand side and take the square root of both sides of the inequality to obtain

$$4(ab + 1)(ac + 1)(bc + 1) \leq (2abc + 3c - a - b)^2.$$

We expand this inequality to get

$$8b^2ac + 8a^2bc + 4ab + 4ac + 4bc + 4 \leq 8c^2ab + (3c - a - b)^2$$

and we move on to prove that this holds. By Lemma 2, we have $c \geq a + b + 2r$. We must check that the inequality

$$4ab + 4ac + 4bc + 4 \leq 16rabc + (3c - a - b)^2.$$

Although, it is easy to see that

$$4ab + 4ac + 4bc + 4 \leq 4abc + 4abc + 4abc + 4abc \leq 16rabc + (3c - a - b)^2.$$

□

The following two lemmas are critical in the proof of Theorem 1. They will be used frequently, in particular, the inequality $b > 3a$ will be utilized (see page 6 in [3]).

Lemma 4. *Let* $\{a,b,c,d,e\}$ *be a Diophantine quintuple with* $a < b < c < d < e$. *Then* $b > 3a$. *Moreover, if* $c > a + b + 2\sqrt{ab} + 1$ *then* $b > max\left\{24a, 2a^{\frac{3}{2}}\right\}$.

Lemma 5. *Let* $\{a,b,c,d\}$ *be a Diophantine quadruple with* $a < b < c < d$. *If* $b < 2a$ *and* $c \geq 9.864b^2$ *or* $2a \leq b \leq 12a$ *and* $c \geq 4.321b^4$, *or* $b > 12a$ *and* $c \geq 721.8b^4$, *then* $d = d_+$.

By combining Lemma 4 and 5, we obtain the following Lemma 6.
Lemma 6. *Let* $\{a,b,c,d,e\}$ *be a Diophantine quintuple with* $a < b < c < d < e$. *Then we have* $ac < 180.45b^3$.

Lemma 7. *Provided that* $\{a,b,c,d,e\}$ *is a Diophantine quintuple with* $a < b < c < d < e$, *then* $b \geq 15, c \geq 24$ *and* $d \geq 1520$.

Proof. Only Diophantine quintuples are considered such that $b > 3a$ and $(a,b) \neq (k, 4k \pm 4)$. (See lemma 6 in [3])

Firstly, take note that if $r > 15$ then $b > 15$. When searching for all Diophantine pairs $\{a,b\}$ such that $2 \leq r \leq 15$ we find that a minimal b can be found from the Diophantine pair $\{1,15\}$. We determine that the smallest c is $1 + 15 + 2\sqrt{1 \cdot 15 + 1} = 24$ because $a + b + 2r$ strictly increases with a. Hence, the smallest Diophantine triple $\{1,15,24\}$ may be extended to a Diophantine quintuple. Since any Diophantine triple $\{a,b,c\}$ that can be extended to a Diophantine quintuple $\{a,b,c,d,e\}$ with $a < b < c < d < e$ satisfies $d = d_+(a,b,c)$ (see Lemma 1), we deduce $d \geq d_+(1,15,24) = 1520$.

□

3. An Operator on Diophantine Triples

This section will explore into Diophantine triples and the ∂-operator. We will start by defining Euler triples in terms of d_-.

Proposition 1. *The Diophantine triple* $\{a,b,c\}$ *is a Euler triple if and only if* $d_-(a,b,c) = 0$.

Proof. Let $\{a,b,c\}$ be a Euler triple, then $c = a + b + 2r$, where $r = \sqrt{ab+1}$. We have

$$\sqrt{ac+1} = a + r \text{ and } \sqrt{bc+1} = b + r.$$

Utilizing these identities, we obtain

$$\begin{aligned} d_-(a,b,c) &= a + b + c + 2abc - 2\sqrt{(ab+1)(ac+1)(bc+1)} \\ &= 2a + 2b + 2r + 2ab(a + b + 2r) - 2r(a + r)(b + r) \\ &= 2a + 2b + 2r + 2a^2b + 2ab^2 + 4abr - 2r(2ab + ar + br + 1) \\ &= 2a + 2b + 2r + 2a^2b + 2ab^2 - 2a(ab+1) - 2b(ab+1) - 2r = 0 \end{aligned}$$

Alternatively, presuming that $d_-(a,b,c) = 0$ implies the following

$$(a+b+c+2abc)^2 = 4(ab+1)(ac+1)(bc+1).$$

Upon expansion of this and further simplifying we arrive at the following equation:

$$a^2+b^2+c^2-2ab-2ac-2bc = 4.$$

Additional manipulations of this gives a result of

$$(c-(a+b))^2 = 4(ab+1) = 4r^2$$

and finally, we get

$$c = a+b \pm 2r.$$

As $c > b > a$, the $'-'$ can be left out so we get $c = a+b+2r$, therefore it is proved that $\{a, b, c\}$ is an Euler triple.

□

Proposition 2. *Let* $\{a, b, c\}$ *be a Diophantine triple with* $c = max\{a, b, c\}$. *We have*

$$a = d_-\big(b, c, d_+(a, b, c)\big) \qquad \textbf{(1)}$$
$$b = d_-\big(a, c, d_+(a, b, c)\big) \qquad \textbf{(2)}$$
$$c = d_-\big(a, b, d_+(a, b, c)\big) \qquad \textbf{(3)}$$

Provided $\{a, b, c\}$ *is not an Euler triple, we have* $c = d_+\big(a, b, d_-(a, b, c)\big)$. *In particular,* $\{a, b, d_{-1}(a, b, c), c\}$ *is a regular Diophantine quadruple.*

Proof. Consider $d_+(a, b, x)\colon \mathbb{R}^+ \to \mathbb{R}^+$ as a function with a fixed a and b. Now consider the following equation where x is not known and a fixed $y \in \mathbb{R}^+ \to \mathbb{R}^+$ thereby $y > \max\{a, b, x\}$:

$$y = a+b+x+2abx+2\sqrt{(ab+1)(ax+1)(bx+1)}.$$

By manipulating this, a quadratic equation is formed:

$$(a+b+x+2abx-y)^2 = 4(ab+1)(ax+1)(bx+1).$$

The two solutions when solving for x are:

$$x = a+b+y+2aby+2\sqrt{(ab+1)(ay+1)(by+1)}$$
$$x = a+b+y+2aby-2\sqrt{(ab+1)(ay+1)(by+1)}$$

We assumed that $y > x$, so the first solution can be discarded. The second solution may be rewritten as

$$x = d_-(a, b, y) = d_-\big(a, b, d_+(a, b, x)\big).$$

One can see that the formulas for d_+ and d_- are symmetric in a, b and c. Since this is the case, we obtain the formulas **(1)**, **(2)** and **(3)**. The fourth formula can be obtained similarly to the previous ones. $\{a, b, d_{-1}(a, b, c), c\}$ being a regular Diophantine quadruple is directly consequential from the formula $c = d_-\big(a, b, d_+(a, b, c)\big)$.

Take note that we have $c = d_+ = d_+\big(a, b, d_-(a, b, c)\big)$.

□

Lemma 8. *Let* $\{a, b, c\}$ *be a Diophantine triple, then* $d_-(a, b, c) \neq a, b, c$.

Proof. We assume that $a < b < c$, without loss of generality. First, start by claiming that $2rst > 2abc + a + b$. This shows that $d_-(a, b, c) < c$, thus $d_-(a, b, c) \neq c$. Verification of this claim will be carried out by showing that

$$\begin{aligned} &4(ab+1)(ac+1)(bc+1) \\ &\qquad = 4a^2b^2c^2 + 4a^2bc + 4ab^2c + 4abc^2 + 4ab + 4ac + 4bc + 4 \\ &\qquad > (2abc + a + b)^2 = 4a^2b^2c^2 + 4a^2bc + 4ab^2c + a^2 + b^2 + 2ab. \end{aligned}$$

From this we must show that

$$4abc^2 + 2ab + 4ac + 4bc + 4 > a^2 + b^2$$

which can be noticed easily.

Now consider the case where $d_-(a, b, c) = a$. This is seen as a quadratic equation in c. The equation is equal to

$$b + c + 2abc = 2\sqrt{(ab+1)(ac+1)(bc+1)}.$$

Solving for c gives the following solution

$$c = a + b \pm \sqrt{4ab + 4 + 4a}.$$

As per the first assumption i.e. $c > b > a$, we may assume that

$$c = a + b + \sqrt{4ab + 4 + 4a} > a + b + 2r.$$

Then by Lemma 2, we have

$$c = a + b + \sqrt{4ab + 4 + 4a} > 4ab.$$

After manipulating this we obtain the inequality below.

$$\begin{gathered} 4ab + 4 + 4a > (4ab - a - b)^2 \\ = 16a^2b^2 - 8a^2b - 8ab^2 + a^2 + b^2 + 2ab > a^2 + b^2 + 2ab. \ \mathbf{(4)} \end{gathered}$$

We know $b > 3a$ by Lemma 2, so we get

$$4ab + 4 + 4a > a^2 + 3ab + 2ab.$$

Hence $4a + 4 > 4a^2$ and this gives $a = 1$. When putting $a = 1$ into the inequality **(4)**, we obtain

$$4b + 4 > 1 + b^2 + 2b$$

giving us a contradiction unless $b < 4$, however due to Lemma 7 the assumption is that $b \geq 15$. Similarly, $b = d_-(a, b, c)$ is impossible.

□

We may add d_+ to any Diophantine triple $\{a, b, c\}$ to obtain a regular Diophantine quadruple $\{a, b, c, d_+\}$. Particularly, three new Diophantine triples $\{a, b, d_+\}$, $\{a, c, d_+\}$ and $\{b, c, d_+\}$ in relation to $\{a, b, c\}$ can be obtained from the triple $\{a, b, c\}$. We may consider these three new Diophantine triples to be closer to an original triple $\{a, b, c\}$ rather than a Euler triple. Let us reverse this such that given a non-Euler triple $\{a, b, c\}$ we want a new Diophantine triple $\{a', b', c'\}$ that is closer to having the property of an Euler triple. To specify this, we introduce the ∂-operator.

Definition 1. *The operator ∂ sends a non-Euler triple $\{a, b, c\}$ to a Diophantine triple $\{a', b', c'\}$ such that*

$$\partial(\{a, b, c\}) = \{a, b, c, d_-(a, b, c)\} - \{max(a, b, c)\},$$

where $\{a, b, c, d_-(a, b, c)\} - \{max(a, b, c)\}$ denotes the set obtained by removing the maximal element from $\{a, b, c, d_-(a, b, c)\}$. For a positive integer D, the operator defined ∂_{-D} on Diophantine triples by:

(i) *For any Diophantine triple $\{a, b, c\}$ we define $\partial_0(\{a, b, c\}) = \{a, b, c\}$.*

(ii) *Provided that $\partial_{-(D-1)}(\{a, b, c\})$ is not an Euler triple, we recursively define $\partial_{-D}(\{a, b, c\}) = \partial(\partial_{-(D-1)}(\{a, b, c\}))$ for $D \geq 1$.*

Additionally, we put $d_{-D}(\{a, b, c\}) = d_-(\partial_{-D+1}(\{a, b, c\}))$.

Particularly, we have $\partial = \partial_{-1}$ and $\partial_{-2}(\{a, b, c\}) = \partial(\partial_{-1}(\{a, b, c\}))$.
The ∂-operator is well defined due to Lemma 8, i.e. a Diophantine triple $\{a, b, c\}$ is mapped to another Diophantine triple unless $\{a, b, c\}$ is not a Euler triple.

Proposition 3. *For any fixed Diophantine triple $\{a, b, c\}$, there exists a unique nonnegative integer $D < \frac{log\ (abc)}{log\ (12)}$ such that $d_{-(D+1)}(a, b, c) = 0$.*

Proof. Proposition 1 shows that if $\{a, b, c\}$ is a Euler triple, then this result is reached.

Now, assume that $\{a, b, c\}$ is not a Euler triple. Through Proposition 2 we know that $\{a, b, d_{-1}(a, b, c), c\}$ is regular Diophantine quadruple. From Lemma 3 we determine that $c > 4ab \cdot d_{-1}(a, b, c)$. Particularly $ab \cdot d_{-1}(a, b, c) < \frac{c}{4} < \frac{abc}{12}$. Note: $ab \geq 3$. The implication of this is that the multiplication of $a'b'c'$ of the elements of the corresponding triple $(a', b', c') \coloneqq \partial_{-k}(\{a, b, c\})$ is less than $\frac{abc}{12^k}$. The is only true if the previous $k - 1$ images were not Euler triples. Therefore, there exists some positive integer $D < \frac{\log\ (abc)}{\log\ (12)}$ where $\{a'', b'', c''\} \coloneqq \partial_{-D}(\{a, b, c\})$ is a Euler triple. Through Proposition 1 we have $d_{-(D+1)}(a, b, c) = 0$.

D is unique due to the fact that the product $a'b'c'$ with $\{a',b',c'\} \coloneqq \partial_{-k}(\{a,b,c\})$ is decreasing with k until we arrive at a Euler triple.

□

Definition 2. *A Diophantine triple $\{a,b,c\}$ is of degree D and is generated by an Euler triple $\{a',b',c'\}$, if $d_{-(D+1)}(a,b,c) = 0$ and $\partial_{-D}(\{a,b,c\}) = \{a',b',c'\}$. If the triple $\{a,b,c\}$ is of degree D we simply write $deg(a,b,c) = D$.*

Remark 1. Note that in the definition the triple $\{a',b',c'\}$ is a Euler triple due to Proposition 1 since $d_-(a',b',c') = 0$ by assumption.

4. System of Pell Equations

Definition 3. *A Pell's equation is any Diophantine equation of the form*

$$ny^2 + 1 = x^2$$

where n is a given positive non-square integer and x and y are integer solutions to be found.

Let $\{a,b,c\}$ be a Diophantine triple with $a < b < c$, and r,s,t be positive integers such that

$$ab + 1 = r^2$$
$$ac + 1 = s^2$$
$$bc + 1 = t^2$$

Assume that $\{a,b,c,d,e\}$ is a Diophantine quintuple with $a < b < c < d < e$, and take

$$ad + 1 = x^2$$
$$bd + 1 = y^2$$
$$cd + 1 = z^2$$

with $x,y,z \in \mathbb{Z}$. Then integers X,Y,Z,W exist such that

$$ae + 1 = X^2$$
$$be + 1 = Y^2$$
$$ce + 1 = Z^2$$
$$de + 1 = W^2$$

We may assume that $d = d_+$ is fixed due to Fujita's results [4], then we have

$$x = at + rs$$
$$y = bs + rt$$
$$z = cr + st$$

By eliminating e from the equations above, the following system of Pell equations can be found:

$$aY^2 - bX^2 = a - b$$
$$aZ^2 - cX^2 = a - c$$
$$bZ^2 - cY^2 = b - c$$
$$aW^2 - dX^2 = a - d$$
$$bW^2 - dY^2 = b - d$$
$$cW^2 - dZ^2 = c - d$$

Lemma 9. *Every integer solution to a Pell equation of the form*

$$aY^2 - bX^2 = a - b$$

with $ab + 1 = r^2$ is obtained from

$$Y\sqrt{a} + X\sqrt{b} = \left(y_0\sqrt{a} + x_0\sqrt{b}\right)\left(r + \sqrt{ab}\right)^n$$

where n, x_0 and y_0 are integers such that $n \geq 0$,

$$1 \leq x_0 \leq \sqrt{\frac{a(b-a)}{2(r-1)}}$$

and

$$1 \leq |y_0| \leq \sqrt{\frac{(r-1)(b-a)}{2a}}$$

5. Linear Forms in Logarithms

Let Λ be a linear form in logarithms of N multiplicatively independent totally real algebraic numbers $\alpha_1, \dots, \alpha_N$ with rational integer coefficients $b_1, \dots, b_N$ such that $b_N \neq 0$.

Definition 4. $\Lambda_1 = 2hlog(\alpha_1) - 2jlog(\alpha_2) + log(\alpha_3)$ where $\alpha_1 = r + \sqrt{ab}$, $\alpha_2 = s + \sqrt{ac}$, $\alpha_3 = \frac{\sqrt{c}(\sqrt{a}+\sqrt{b})}{\sqrt{b}(\sqrt{a}+\sqrt{c})}$ (see Lemma 26 of [3]).

This definition is important for the proofs later on in the paper.

Proposition 4. *If {a, b, c, d, e} is a Diophantine quintuple with a < b < c < d < e, then we have $ac < 6.77 \cdot 10^{25}$, $d < 1.83 \cdot 10^{52}$. Provided that $c > 2 \cdot 10^8$, we have*

$$h < 2.8376 \cdot 10^{10} log(\alpha_2) \log(c) < 5.136 \cdot 10^{13}.$$

6. Euler Triples

In this section, we will continue to assume that $\{a, b, c, d, e\}$ is a Diophantine quintuple with $a < b < c < d < e$ while also assuming that $\{a, b, c, d\}$ is an Euler quadruple of the form $\{a, b, a + b + 2r, 4r(a + r)(b + r)\}$.

Lemma 10. If $n \equiv -\varepsilon r \ (mod\ st)$, then we have $r < 900154$ and $h < 9.6 \cdot 10^{15}$.

Proof. This can be found in He, Togbé and Ziegler's paper (cf. Lemma 25 in [3])

□

We may assume that $\{a, b, c, d, e\}$ is Diophantine quintuple such that $a < b < c < d < e$ and $\{a, b, c\}$ is a Euler triple. Then, by Lemma 21 and Lemma 23-25 in He, Togbé and Ziegler's paper [3], we have

$$r < 900154 \text{ and } h < 1.9 \cdot 10^{16}.$$

Lemma 11. *Assume that* M *is a positive integer. Let* p/q *be the convergent of the continued fraction expansion of a real number* κ *such that* $q > 6M$ *and let*

$$\eta = \|\mu q\| - M \cdot \|\kappa q\|$$

where $\|\cdot\|$ *denotes the distance from the nearest integer. If* $\eta > 0$*, then the inequality*

$$0 < J_\kappa - K + \mu < AB^{-J}$$

has no solutions in integers J *and* K *with*

$$\frac{log\left(\frac{A_q}{\eta}\right)}{\log(B)} \le J \le M$$

We apply Lemma 11 to

$$\Lambda_1 = 2hlog\left(r + \sqrt{ab}\right) - 2jlog\left(s + \sqrt{ac}\right) + log\left(\frac{\sqrt{c}\left(\sqrt{a} + \sqrt{b}\right)}{\sqrt{b}\left(\sqrt{a} + \sqrt{c}\right)}\right)$$

with $s = a + r$, $c = a + b + 2r$ and

$$\kappa = \frac{log\left(r+\sqrt{ab}\right)}{log\left(s+\sqrt{ac}\right)} \quad , \quad \mu = \frac{log\left(\frac{\sqrt{c}\left(\sqrt{a}+\sqrt{b}\right)}{\sqrt{b}\left(\sqrt{a}+\sqrt{c}\right)}\right)}{log\left(s+\sqrt{ac}\right)} \quad , \quad A = \frac{1}{log\left(s+\sqrt{ac}\right)} \quad , \quad B = \left(r + \sqrt{ab}\right)^2$$

and $J = 2h$, $M = 1.9 \cdot 10^{16}$. He, Togbé and Ziegler ran a GP program to check all 58258307 pairs (a, b) such that $2 \le r \le 900153$. For all cases, $J \le 15$ was obtained, which contradicts $J = 2h \ge 4c(r - 1) \ge 48$.

Theorem 3. *A Euler triple* $\{a, b, a + b + 2\sqrt{ab + 1}\}$ *cannot be extended to a Diophantine quintuple.*

7. Non-Euler Triples

In this section, non-Euler triples will be explored and two cases will be considered: one case of a triple with a degree of one and the other case of a triple with a degree greater than one. We will start with the first case where it has a degree of one.

Theorem 4. *A Diophantine triple* $\{a, b, c\}$ *cannot be extended to a Diophantine quintuple if* $deg(a, b, c) = 1$.

Proof. Assume $a < b < c$. If $\deg(a, b, c) = 1$, then $\{d_{-1}, a, b\}$ is a Euler triple, where

$$d_{-1} = d_{-}(a, b, c).$$

The quadruple $\{d_{-1}, a, b, c\}$ is regular due to Proposition 2. Thus, we have

$$d_{-1} = a + b \pm 2r$$

and

$$c = d_{+}(a, d_{-1}, b) = 4r(r \pm a)(b \pm r)$$

Assuming $b > 10000$, then $d_{-1} \geq a + b - 2\sqrt{\frac{b^2}{24} + 1} > 0.59b$ with $c > abd_{-1}$ and we get

$$2.36(ab)^2 < 4a^2bd_{-1} < ac < 6.77 \cdot 10^{25}$$

Hence, $ab < 5.36 \cdot 10^{12}$ and $r \leq 2315167$ is obtained. Since $b > max\left\{24a, 2a^{\frac{3}{2}}\right\}$ due to lemma 4, we have $a < \left(\frac{r^2}{2}\right)^{\frac{2}{5}} < 93596$.

We apply Lemma 11 to Λ_1 and check 109748916 pairs (a, r) such that

$$b = \frac{r^2 - 1}{a} \quad , \quad c = 4r(r \pm a)\,(b \pm r)$$

and κ, μ, A, B are taken as in the previous section.

Furthermore, we take $J = 2h$ and $M = 1.9 \cdot 10^{16}$. It took 16 hours to run He, Tobgé and Ziegler's GP program. For all 219498932 cases, there is $J \leq 15$, which contradicts the fact that $J = 2h > 10\sqrt{ac} > 20\sqrt{2}$.

□

Next, the case of the triple with a degree greater than one will be looked at.

Theorem 5. *A Diophantine triple* $\{a, b, c\}$ *cannot be extended to a Diophantine quintuple if* $deg(a, b, c) \geq 2$.

Proof. Consider a Diophantine triple $\{a, b, c\}$ with $deg(a, b, c) \geq 2$. By Proposition 4, we assume that $ac < 6.77 \cdot 10^{25}$. Since $deg(a, b, c) \geq 2$, there exists positive integers d_{-1} and d_{-2} such that

$$d_{-1} = d_{-}(a,b,c)$$
$$d_{-2} = d_{-}(a,b,d_{-1})$$

Since $\{a,b,c\}$ is not a Euler triple, we have $c > a + b + 2r$ where $r = \sqrt{ab+1}$ and $c > ab$ (by Lemma 1). We also have $ac < 180.45b^3$ (by Lemma 5). This gives

$$4ab < c < 180.45\frac{b^3}{a}.$$

The remainder of the proof will split the interval $(4ab, 180.45\frac{b^3}{a})$ into five subintervals:

$$c \in \left(4ab, 4a^{\frac{1}{2}}b^{\frac{3}{2}}\right] \cup \left(4a^{\frac{1}{2}}b^{\frac{3}{2}}, 4ab^2\right] \cup \left(4ab^2, 4a^{\frac{3}{2}}b^{\frac{5}{2}}\right] \cup \left(4a^{\frac{3}{2}}b^{\frac{5}{2}}, 4a^2b^3\right]$$
$$\cup \left(4a^2b^3, \frac{180.45b^3}{a}\right]$$

Note: the last interval $\left(4a^2b^3, \frac{180.45b^3}{a}\right]$ is not empty if and only if $1 \le a \le 3$.

Case 1:

We have the interval $4ab < c \le 4a^{\frac{1}{2}}b^{\frac{3}{2}}$. Since $c = d_{+}(a,b,d_{-1})$, we have $c > 4abd_{-1}$. From this, we have

$$ad_{-1} < \frac{c}{4b} < (ab)^{\frac{1}{2}}$$

in which we can obtain $ab > (ad_{-1})^2$. Since $c > 4abd_{-1}$, we get $ac > 4(ab)(ad_{-1})$ and

$$ad_{-1} < \left(\frac{ac}{4}\right)^{\frac{1}{3}} < \left(\frac{6.77\cdot 10^{25}}{4}\right)^{\frac{1}{3}} < 256749472.$$

Let $r_{(a,d_{-1})} = \sqrt{ad_{-1}+1}$. Since $ad_{-1}+1$ is a perfect square, $r_{(a,d_{-1})}$ is a positive integer with $2 \le r_{(a,d_{-1})} \le 16023$. He, Togbé and Ziegler ran a short GP program to find the number of pairs (a, d_{-1}) to be checked in this range (see page 38 in [3]). Notice that $\{a, d_{-1}, b\}$ is a Diophantine triple.

For a fixed pair (a, d_{-1}), positive integers $U = r = \sqrt{ab+1}$ and $V = \sqrt{bd_{-1}+1}$ exist such that $b = \frac{V^2-1}{d_{-1}} = \frac{U^2-1}{a}$ and $max\{U,V\} \le b^{\frac{1}{2}}$. We have

$$4a^2b < a\cdot 4ab < ac < 6.77\cdot 10^{25}$$

and now may assume that $max\{U,V\} \le 4.12\cdot 10^{12}$.

To find all possible values of b, the following Pell equation is used:

$$\mathcal{A}V - \mathcal{B}U^2 = \mathcal{A} - \mathcal{B} \ (5)$$

where $\mathcal{AB}+1 = \mathcal{R}^2$, $\mathcal{A} < \mathcal{B}$ and $0 < \mathcal{A},\mathcal{B},\mathcal{R} \in \mathbb{Z}$. All positive integer solutions to (5) can be obtained by (Lemma 8):

$$\mathcal{V}\sqrt{\mathcal{A}}+\mathcal{U}\sqrt{\mathcal{B}}=\mathcal{V}_q\sqrt{\mathcal{A}}+\mathcal{U}_q\sqrt{\mathcal{B}}=\left(\mathcal{V}_0\sqrt{\mathcal{A}}+\mathcal{U}_0\sqrt{\mathcal{B}}\right)\left(\mathcal{R}+\sqrt{\mathcal{A}\mathcal{B}}\right)^q$$

with $q\geq 0$, where $(\mathcal{V}_0,\mathcal{U}_0)$ is a fundamental solution to (5). This satisfies

$$0\leq|\mathcal{V}_0|\leq\sqrt{\frac{1}{2}\mathcal{A}(\mathcal{B}-\mathcal{A})(\mathcal{R}-1)}$$

and

$$0<\mathcal{U}_0\leq\sqrt{\frac{\mathcal{A}(\mathcal{B}-\mathcal{A})}{2(\mathcal{R}-1)}}$$

All $r_{(a,d_{-1})}$ in the range from 2 to 16023 can be operated in the GP program.

For each $\mathcal{R}=r_{(a,d_{-1})}$, consider the divisors d' of $\mathcal{R}^2-1$ with $1\leq d'\leq\mathcal{R}$ and let $\mathcal{A}=d'$, $\mathcal{B}=(\mathcal{R}^2-1)/\mathcal{A}$. For each pair $(\mathcal{A},\mathcal{B})$, all possible fundamental solutions $(\mathcal{V}_0,\mathcal{U}_0)$ to equation (5) can be found while considering the corresponding sequences $\mathcal{U}_q$. Note that not all solutions $\mathcal{U}$ of (5) satisfy $\mathcal{A}|(\mathcal{U}^2-1)$. If $\mathcal{A}|(\mathcal{U}^2-1)$ and $\mathcal{U}=\mathcal{U}_q<4.12\cdot 10^{12}$, then (a,d_{-1},b) or $(d_{-1},a,b)=(\mathcal{A},\mathcal{B},\mathcal{C})$ and $c=d_+(a,d_{-1},b)$ where $\mathcal{C}=\frac{\mathcal{U}^2-1}{\mathcal{A}}$.

By applying Lemma 11 to Λ_1, all 2340242 triples (a,b,c) can be checked in 15 minutes using the GP program. For all cases, $J\leq 6$ is obtained, however this is impossible as $J>20\sqrt{2}$.

Case 2:

We have the interval $4a^{\frac{1}{2}}b^{\frac{3}{2}}<c\leq 4ab^2$. Since $c=d_+(a,b,d_{-1})$, by assumption we get $d_{-1}<\frac{c}{4ab}<b$. This gives $b=\max(a,b,d_{-1})$ and Lemma 3 yields $c<4b(ad_{-1}+1)$. Conversely, by assumption we have that $4a^{\frac{1}{2}}b^{\frac{3}{2}}<c$ and we get $(ab)^{\frac{1}{2}}-1<ad_{-1}$. Hence, we get

$$d_{-2}=d_-(a,b,d_{-1})<\frac{b}{4ad_{-1}}<\frac{b}{4\left((ab)^{\frac{1}{2}}-1\right)}$$

and

$$4ad_{-2}<\frac{ab}{(ab)^{\frac{1}{2}}-1}<(ab)^{\frac{1}{2}}+2$$

From $(4ad_{-2}-2)^2<ab<(ad_{-1}+1)^2$, we have $4ad_{-2}<ad_{-1}+3$ and so $d_{-2}<d_{-1}$. By substituting $ab>(4ad_{-2}-2)^2$ into $ac>4(ab)(ad_{-1})>4(ab)(ad_{-2})$, we obtain

$$4(4ad_{-2}-2)^2(ad_{-2})<ac<6.77\cdot 10^{25}.$$

Therefore, $ad_{-2}<101891096$ and we get $r_{(a,d_{-2})}=\sqrt{ad_{-2}+1}\leq 10095$. Furthermore, we know that

$$d_{-1} < b < \left(\frac{6.77 \cdot 10^{25}}{4}\right)^{\frac{2}{3}} < 6.6 \cdot 10^{16}$$

Analogously to Case 1, we put (a, d_{-2}) or $(d_{-2}, a) = (\mathcal{A}, \mathcal{B})$ in equation (5). We set (a, d_{-2}, d_{-1}) or $(d_{-2}, a, d_{-1}) = (\mathcal{A}, \mathcal{B}, \frac{\mathcal{U}_q{}^2-1}{\mathcal{A}})$ when $\mathcal{A} | \mathcal{U}_q{}^2 - 1$, for $\mathcal{U}_q < b^{\frac{1}{2}} < 2.57 \cdot 10^8$. By applying Lemma 11 to Λ_1, all 2565234 triples (a, b, c) can be checked in 20 minutes using $b = d_+(a, d_{-1}, d_{-2})$ and $c = d_+(a, b, d_{-1})$. For all cases, $J \leq 6$ was obtained which is impossible as $J > 20\sqrt{2}$.

Case 3:

We have the interval $4ab^2 < c \leq 4a^{\frac{3}{2}}b^{\frac{5}{2}}$.

We see that the inequality $4a^2b^2 < ac < 6.77 \cdot 10^{25}$ yields the upper bound $r < 2028300$. Since $b^{\frac{1}{2}} < \sqrt{ab+1} = r$, we have an upper bound for $b^{\frac{1}{2}}$.

If we assume that $b > d_{-1}$, then $b = d_+(a, d_{-1}, d_{-2}) > 4ad_{-1}$ and this would give $d_{-1} < \frac{b}{4a}$. However, this yields

$$4ab^2 < c < 4abd_{-1} + 4b < b^2 + 4b$$

which is impossible. As a result, we may assume that $b < d_{-1}$. Since $d_{-1} < \frac{c}{4ab}$, we get

$$d_{-1} < a^{\frac{1}{2}}b^{\frac{3}{2}}$$

and thus,

$$ad_{-2} < \frac{d_{-1}}{4b} < \frac{(ab)^{\frac{1}{2}}}{4} < \frac{r}{4} < 507075$$

If $r_{(a,d_{-2})} = \sqrt{ad_{-2}+1}$, then we have $r_{(a,d_{-2})} \leq 712$. Similar to Case 1, we set (a, d_{-2}) or $(d_{-2}, a) = (\mathcal{A}, \mathcal{B})$ in equation (5). We set (a, d_{-2}, b) or $(d_{-2}, a, b) = (\mathcal{A}, \mathcal{B}, \frac{\mathcal{U}_q{}^2-1}{\mathcal{A}})$ when $\mathcal{A} | \mathcal{U}_q{}^2 - 1$, for $\mathcal{U}_q < b^{\frac{1}{2}} < 2028300$.

We apply Lemma 11 to Λ_1 by using $d_{-1} = d_+(a, d_{-2}, b)$ and $c = d_+(a, b, d_{-1})$ to check all 102032 triples (a, b, c) in 1 minute and 15 seconds using the GP program mentioned previously. For all cases, $J \leq 14$ was obtained, which contradicts $J > 20\sqrt{2}$.

Case 4:

We have the interval $4a^{\frac{3}{2}}b^{\frac{5}{2}} < c \leq 4a^2b^3$. Similar to Case 3, we may assume that $b < d_{-1}$. We have

$$d_{-1} < \frac{c}{4ab} < ab^2 \quad \text{and} \quad d_{-2} < \frac{d_{-1}}{4ab} < \frac{b}{4}$$

The implication of these inequalities is that $\{a, d_{-2}, b\}$ is not a Euler triple. Thus, a positive integer exists

$$d_{-3} = d_{-}(a, d_{-2}, b)$$

Now, we estimate the upper bound of ad_{-3}. By applying Lemma 3 to the Diophantine quadruple $\{a, b, d_{-1}, c\}$, we have $c < 4abd_{-1} + 4d_{-1}$. This implies that

$$d_{-1} > \frac{c}{4ab+4} > \frac{a^{\frac{3}{2}}b^{\frac{5}{2}}}{ab+1}$$

The Diophantine quadruple $\{a, d_{-2}, b, d_{-1}\}$ provides $d_{-1} < 4abd_{-2} + 4b$, so

$$d_{-2} > \frac{d_{-1} - 4b}{4ab}$$

With $b > 4ad_{-2}d_{-3}$, we have

$$ad_{-3} < \frac{b}{4d_{-2}} < \frac{ab^2}{d_{-1} - 4b} < \frac{ab}{\frac{(ab)^{\frac{3}{2}}}{ab+1} - 4}$$

Conversely, $4(ab)^{\frac{5}{2}} < ac < 6.77 \cdot 10^{25}$ yields $ab < 1.24 \cdot 10^{10}$. Therefore, we obtain

$$ad_{-3} < 111360$$

Let $r_{(a,d_{-3})} = \sqrt{ad_{-3} + 1}$, then we get $r_{(a,d_{-3})} \leq 333$. There are 8854 pairs (a, d_{-3}) that satisfy this inequality. Put (a, d_{-3}) or $(d_{-3}, a) = (\mathcal{A}, \mathcal{B})$ into equation (5). To obtain solutions for this equation, we put (a, d_{-3}, d_{-2}) or $(d_{-3}, a, d_{-2}) = (\mathcal{A}, \mathcal{B}, \frac{\mathcal{U}_q{}^2-1}{\mathcal{A}})$ if $\mathcal{A} | \ \mathcal{U}_q{}^2 - 1$.

Since $b \leq ab < 1.24 \cdot 10^{10}$, we may assume $\mathcal{U}_q < b^{\frac{1}{2}} < 111356$. Additionally, we compute $b = d_{-}(a, d_{-3}, d_{-2})$, $d_{-1} = d_{+}(a, d_{-2}, b)$ and $c = d_{+}(a, b, d_{-1})$ and apply Lemma 11 to Λ_1. All 36762 triples (a, b, c) are checked in 26 seconds using the GP program and $J \leq 6$ was obtained for all cases, which contradicts the fact that $J > 20\sqrt{2}$.

Case 5:

We have the interval $4a^2b^3 < c \leq \frac{180.45b^3}{a}$. This is not empty if and only if $1 \leq a \leq 3$. If it is nonempty, we have $4a^3 < 180.45$. Evidently, $b < d_{-1}$. Using Lemma 3, we have

$$\frac{a^2b^3}{ab+1} < \frac{c}{4(ab+1)} < d_{-1} < \frac{c}{4ab} < \frac{45.2b^2}{a^2} \qquad (6)$$

We have $d_{-2} = d_{-}(a, b, d_{-1}) < \frac{d_{-1}}{4ab} < \frac{11.3b}{a^3}$.

If the Diophantine triple $\{a, b, d_{-2}\}$ is not a Euler triple, then a positive integer $d_{-3} = d_{-}(a, b, d_{-2})$ exists.

When $b < d_{-2}$, we have $d_{-2} > 4abd_{-3} \geq 12b$. We have $d_{-2} < \frac{11.3b}{a^3}$ and $d_{-2} \geq 12b$ providing $12a^3 < 11.3$, which is impossible.

When $b > d_{-2}$, then $b > 4abd_{-2}d_{-3} \geq 12d_{-2}$. Since $d_{-1} < 4abd_{-2} + 4b$, we have $d_{-2} > \frac{d_{-1}-4b}{4ab}$, so $b > 12d_{-2} \geq \frac{3d_{-1}-12b}{ab}$. This implies that $d_{-1} < \frac{ab^2+12b}{3}$. Combining this with (6), we get

$$\frac{a^2b^3}{ab+1} < \frac{ab^2+12b}{3}$$

As a result, we get $2a^2b^2 - 13ab - 12 < 0$. We have $ab < 8$ which leads to $\{a, b\} = \{1, 3\}$ or $\{2, 4\}$. However, there is no integer d_2 that is less than b such that $\{a, d_{-2}, b\}$ is a Diophantine triple. Thus, the Diophantine triple $\{a, b, d_{-2}\}$ is a Euler triple.

For $1 \leq a \leq 3$ and $r \leq 16023$, we have

$$b = \frac{r^2 - 1}{a}$$

and

$$d_{-2} = a + b \pm r = a + \frac{r^2 - 1}{a} \pm 2r$$

Furthermore, we calculate $d_{-1} = d_+(a, d_{-2}, b)$ and $c = d_+(a, b, d_{-1})$.
All 69428 triples (a, b, c) were checked in 40 seconds, and $J \leq 16$ was obtained for all cases, which contradicts $J > 20\sqrt{2}$ again.

Since all cases are contradictions of $J > 20\sqrt{2}$, we conclude that a Diophantine triple $\{a, b, c\}$ cannot be extended to a Diophantine quintuple if $\deg(a, b, c) \geq 2$.

□

We will now proceed to give an outline of the proof of Theorem 1 using what has been addressed in the paper above.

8. Proof of Theorem 1

Assume that $\{a, b, c, d, e\}$ is a Diophantine quintuple with $a < b < c < d < e$. By Theorem 2 (cf. page 1678-1697 in [2]),

$$\{a, b, c, d\}$$

is a regular quadruple, that is d is solely determined by a, b and c.

By Proposition 3, for some arbitrarily fixed Diophantine triple $\{a, b, c\}$, there exists a positive integer $D = \deg(a, b, c)$ such that an Euler triple $\{a', b', c'\}$ generates $\{a, b, c\}$.

Theorem 3,4 and 5 show that for $D = 0, 1$ and $D \geq 2$ there is no extension from a Diophantine triple to a quintuple $\{a, b, c, d, e\}$.

This concludes the proof of Theorem 1.

□

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