Properties of “Pathological” Functions and their Classes

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Abstract

This paper is meant to serve as an exposition on the theorem proved by Richard Aron, V.I. Gurariy and J.B. Seoane in their 2004 paper [2], which states that ‘The Set of Differentiable but Nowhere Monotone Functions (DNM) is Lineable.’ I begin by showing how certain properties that our intuition generally associates with continuous or differentiable functions can be wrong. This is followed by examples of functions with surprisingly unintuitive properties that have appeared in mathematical literature in relatively recent times. After a brief discussion of some preliminary concepts and tools needed for the rest of the paper, a construction of a ‘Continuous but Nowhere Monotone (CNM)’ function is provided which serves as a first concrete introduction to the so called “pathological functions” in this paper. The existence of such functions leaves one wondering about the ‘Differentiable but Nowhere Monotone (DNM)’ functions. Therefore, the next section offers a description of how the 2004 paper by the three authors shows that such functions are very abundant and not mere fancy counter-examples that can be called “pathological.”

1 Introduction

A wide variety of assumptions about the nature of mathematical objects inform our intuition when we perform calculations. These assumptions have their utility because they help us perform calculations swiftly without being over skeptical or neurotic. The problem, however, is that it is not entirely obvious as to which assumptions are legitimate and can be rigorously justified and which ones are not.

Consider the following statements:

(i) An everywhere continuous function has to be differentiable at least somewhere.

(ii) An everywhere continuous function has to be monotonic at least somewhere.

(iii) An everywhere differentiable function has to be monotonic at least somewhere.

These statements on the surface might seem plausible and might inform our intuition when we perform calculations but surprisingly, all of these statements are incorrect!

The rigorous formulation of Real Analysis in the nineteenth century facilitated the creation of abstract mathematical tools of great power. Subsequently, the researchers in the field have been using these abstract tools to unravel increasingly complex results. In 1872, Weierstrass constructed a function that was everywhere continuous, nowhere differentiable and nowhere monotonic. Thus, the existence of the Weierstrass function refutes the statements (i) and (ii). An approximate graphical depiction of the Weierstrass Function is shown below:

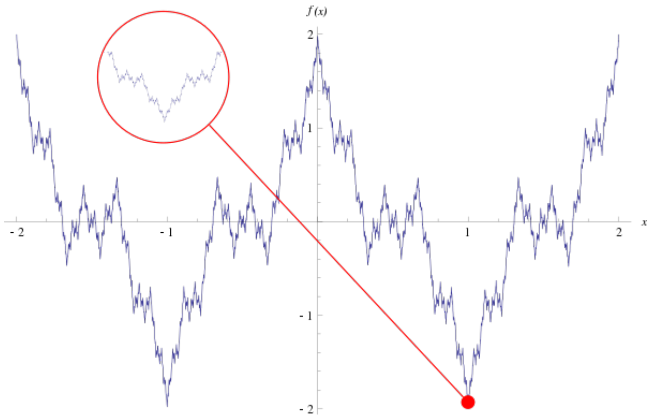

Figure 1: Weierstrass function [3]

Wherever and however much you zoom in to this graph, it will still resemble the whole graph. This function is everywhere continuous, nowhere differentiable, nowhere monotonic and extremely weird. In the years that followed Weierstrass' discovery, a wide variety of such functions with even weirder properties were found by many other mathematicians. So unintuitive and disconcerting were these discoveries, that the great mathematician Henri Poincarè remarked:
"Logic sometimes makes monsters...we have seen a mass of bizarre functions which appear to be forced to resemble as little as possible to honest functions which serve some purpose."
Subsequently, such functions were dubbed 'Pathological'.
Terms like "Pathological" suggest that there is something intrinsically wrong with such functions. They imply that such functions are in some way misfits, or atypical. In reality, these functions are abundant in number and hence quite typical. But before we discuss their abundance, we need to familiarise ourselves with some preliminary concepts.

2 Preliminary Concepts

We usually construct irrational numbers as limits of Cauchy sequences of rational numbers. Similarly, the "pathological" functions are usually approached in this oblique way as limits of sequences of "simpler" functions. Therefore, a careful study of certain properties of sequences of functions becomes important. What follows is a brief account of major definitions and lemmas that will be used later in the paper.

Definition: Pointwise Convergence: *Suppose $\{f_n : n = 0, 1, 2, 3, ...\}$ is a sequence of functions defined on an interval I. We say that $f_n(x)$ converges point-wise to the function $f(x)$ on the interval I if*

$$\forall x \in I : \lim_{n \longrightarrow \infty} f_n(x) = f(x).$$

Definition: Banach Space: *A Banach space is a vector space X over any scalar field K, which is equipped with a norm $||.||_X$ in which every Cauchy sequence converges.*

Example: The only Banach space that we will be dealing with is the Banach space of continuous functions on an interval $I = [a, b]$, denoted as $C[a, b]$. The norm that acts on this space (called 'Supremum Norm') is defined below:

$$||f||_I := \sup_{x \in I} |f(x)|.$$

Definition: Uniform Convergence: *A sequence of functions f_n on an interval I is said to 'converge uniformly' to $f(x)$ on I if*

$$\lim_{n \longrightarrow \infty} ||f_n - f||_I = 0.$$

Lemma: (Uniform Convergence $\Rightarrow$ Point-wise Convergence.)
If the sequence of functions $f_n(x)$ converges uniformly to $f(x)$ on the interval I, then $f_n(x)$ converges pointwise to $f(x)$.

Lemma: (Limit of a uniformly convergent sequence of continuous functions is continuous.)
Suppose $f_n(x)$ is a sequence of continuous functions on an interval I and suppose also that $f_n(x)$ converges uniformly to $f(x)$ on the interval I. Then the limit function $f(x)$ is also continuous.

3 A "Pathological" Construction

We now give an example of a so called "Pathological" function. We wish to construct a sequence of functions iteratively [6]. The function f_0 would serve as the initial seed value of the iteration.

Consider a function $f_0 : [0, 1] \longrightarrow \mathbb{R}$ to be the line segment joining the points $(0, 0)$ and $(1, 1)$. Thus, the function can be thought of as an ordered pair of points:

$$f_0 \equiv ((0, 0), (1, 1)).$$

The graphical representation of the function is shown below:

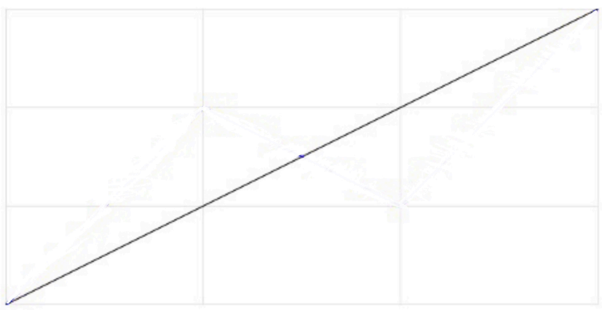

Figure 2: 'Seed Segment' of the iteration. [6]

We now define a recursive function F which maps a segment $((a_1, b_1), (a_2, b_2))$ to a collection of three segments. $F : ((a_1, b_1), (a_2, b_2)) \longmapsto ((a_1, b_1), (\frac{2a_1+a_2}{3}, \frac{b_1+2b_2}{3}), (\frac{a_1+2a_2}{3}, \frac{2b_1+b_2}{3}), (a_2, b_2))$
Using this recursion, we define the next function f_1 as follows:

$$f_1 := F(f_0) \equiv \left((0, 0), \left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, \frac{1}{3}\right), (1, 1)\right)$$

This means that f_1 is a collection of three segments viz.; $((0,0),\left(\frac{1}{3},\frac{2}{3}\right))$, $(\left(\frac{1}{3},\frac{2}{3}\right),\left(\frac{2}{3},\frac{1}{3}\right))$ and $(\left(\frac{2}{3},\frac{1}{3}\right),(1,1))$. Graphical depiction of f_1 is given below.

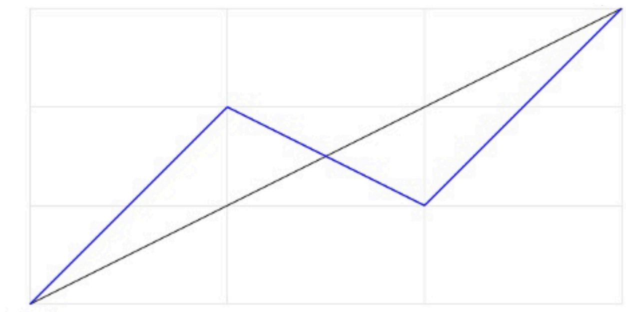

Figure 3: Output after first iteration. [6]

We can now form a sequence of functions $(f_n)_{n\geqslant 0}$ wherein we apply the function F on each of the segments of f_{n-1} to get f_n. Therefore, f_n would be a collection of 3^{n} segments and would be represented as a '$(3^n+1)-$tuple' of points. Each of these functions looks like a *zigzag path.* Each vertex of this path will be called a **"Sharp Point."**
After a few such iterations, the terms would start looking like the following graph:

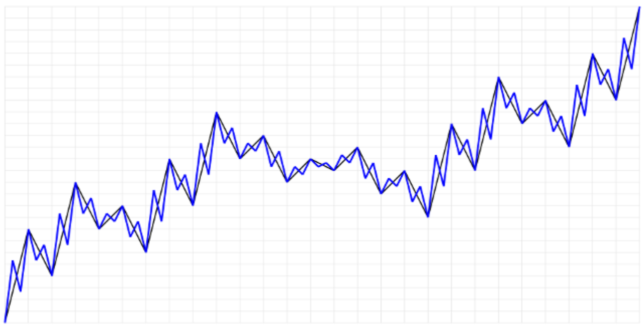

Figure 4: Output after a few iterations. [6]

We are interested in the behaviour of this sequence as n tends to infinity. In particular, we want to prove that this sequence converges, that the limit of the sequence is a continuous function and that it is not monotonic anywhere.

The reader can convince himself of the validity of the following observations which will be crucial for our proof:

- The x-coordinates of sharp points of f_n are $\frac{i}{3^{\mathrm{n}}}$ where i is a natural number such that $0 \leqslant i \leqslant 3^{\mathrm{n}}$.
- The end-points of the segments are invariant under iteration. This means that if $m < n$, then all sharp points of f_m would also be sharp points of f_n i.e. the two functions equate at $x = \frac{i}{3^{\mathrm{m}}}$.
- The first 'non-common sharp point' between f_n and f_m would be the second sharp point of f_n from the left i.e. $x = \frac{1}{3^{\mathrm{n}}}$ and $f_n(\frac{1}{3^{\mathrm{n}}}) = \frac{2^{\mathrm{n}}}{3^{\mathrm{n}}}$. Since, $\frac{1}{3^{\mathrm{n}}} \in [0, \frac{2^{\mathrm{n}}}{3^{\mathrm{m}}}]$ therefore on this interval the equation of the first segment of f_m is $y = \frac{2^{\mathrm{m}}}{3^{2\mathrm{m}}}x$. Thus, $f_m(\frac{1}{3^{\mathrm{n}}}) = \frac{2^{\mathrm{m}}}{3^{\mathrm{n}+2\mathrm{m}}}$.

Equipped with these observations, I now wish to prove that this sequence is Cauchy on the Banach Space $C[0,1]$ with the supremum norm.

Proof. It is simple to see that

$$||f_n - f_m||_{[0,1]} = \sup(|f_n(x) - f_m(x)|) = \max(|f_n(x) - f_m(x)|).$$

Now we know that; f_m and f_n are continuous, both functions equate at the sharp points of f_m and both functions are piece-wise linear. Therefore, $f_n(x) - f_m(x)$ would be maximised at those sharp points of f_n that are not the sharp points of f_m.
Hence,

$$\max(|f_n(x) - f_m(x)|) = \max\left(|f_n(\frac{i}{3^{\mathrm{n}}}) - f_m(\frac{i}{3^{\mathrm{n}}})|\right)$$

where $0 \leqslant i \leqslant 3^{\mathrm{n}}$ and $3^{\mathrm{n-m}} \nmid i$. (To ensure that the 'common sharp points' are excluded)
The seed segment that we chose earlier is part of the line $y = x$. This leads to a symmetry such that $\forall i : 0 \leqslant i \leqslant 3^{\mathrm{n}}$ where i is not a multiple of $3^{\mathrm{n-m}}$, the values $(f_n(\frac{i}{3^{\mathrm{n}}}) - f_m(\frac{i}{3^{\mathrm{n}}}))$ are all equal to each other. Because of this symmetry we just calculate the value of $f_n(x) - f_m(x)$ at the first non-common sharp point i.e. at $i = 1$.
Thus,

$$\max\left(|f_n(\frac{i}{3^{\mathrm{n}}}) - f_m(\frac{i}{3^{\mathrm{n}}})|\right) = f_n(\frac{1}{3^{\mathrm{n}}}) - f_m(\frac{1}{3^{\mathrm{n}}}) = \frac{2^{\mathrm{n}}}{3^{\mathrm{n}}} - \frac{2^{\mathrm{m}}}{3^{\mathrm{n+2m}}}$$

Therefore,

$$||f_n - f_m||_{[a_1,a_2]} = \frac{2^{\mathrm{n}}}{3^{\mathrm{n}}} - \frac{2^{\mathrm{m}}}{3^{\mathrm{n+2m}}}$$

It is now simple to see that:

$$\lim_{\substack{m \longrightarrow \infty \\ n \longrightarrow \infty}} ||f_m - f_n||_{[a_1,a_2]} = 0.$$

The sequence therefore is Cauchy and uniformly converges to some function $f \coloneqq \lim_{n \longrightarrow \infty} f_n$. □

We have shown that f is defined at every point in [0, 1] because uniform continuity implies point-wise continuity and f is continuous because it is a limit of a uniformly convergent sequence of continuous functions.
Moreover, $\forall n \geqslant 0$ and $0 \leqslant m < 3^n$ the function f_{n+1} (and hence f as well) is not monotone on the interval $[\frac{m}{3^n}, \frac{m+1}{3^n}]$. Since any sub-interval of $[0,1]$ contains an interval of this form, therefore, the function f is not monotone anywhere.
Hence we have successfully constructed a function that is everywhere continuous but nowhere monotone (or CNM).

4 Lineability of the set of DNM functions

The previous construction gives us a general flavour of how we approach "pathological" functions. The function in the previous section was shown to be 'Continuous but Nowhere Monotone' i.e. CNM. An obvious question now arises of whether 'Differentiable but Nowhere Monotone' i.e. DNM functions exist. This question was settled by Denjoy who, in his 1915 paper [4] provided the first construction of a DNM function. Subsequently, other such functions were constructed by various Mathematicians.

Our construction of the CNM function in the previous section, although tedious, is comparatively simpler than the constructions by Denjoy and others. Because of the way in which DNM functions are constructed, either no simple graphical representation of these functions exists or, if existent, is not commonly available [1]. The unavailability of such graphical representations is a testament to these functions being extremely unintuitive.

These discoveries prompted mathematicians to ask the questions:

- How abundant are these "Pathological Functions?" and
- Are they sufficiently scarce to be dismissed as mere aberrations and be called "Pathological?"

To tackle such problems, the concept of 'Lineability' and 'Spaceability' were defined by V.I. Gurariy [2]. But before we define these concepts, we need to understand certain prerequisites:

- Let S be a subset of a topological space X. A point $x \in X$ is called a **Limit Point** of S if every neighbourhood of x contains at least one point that lies in S.
- A set is **closed** if and only if it contains all of its limit points. It is called **open** if its complement is closed.
- A vector space, all elements of which cannot be written as linear combinations of finitely many basis elements, is called an **infinite dimensional vector space.**

  Example: The set of all polynomials with real coefficients $\mathbb{R}[x]$ is an infinite dimensional vector space

We are now equipped for a discussion of Lineability and Spaceability.

Definition: Lineability: *A set X of functions is called 'Lineable' if $X \bigcup \{0\}$ contains an infinite dimensional vector-space.*
Example: The set of continuous functions on an interval $C[a, b]$ is lineable because it contains the set of all polynomials restricted on the interval $[a, b]$.

Definition: Spaceability: *A set X of functions is called 'Spaceable' if $X \bigcup \{0\}$ contains a closed infinite-dimensional subspace.*
Example: The set of continuous functions on an interval $C[a, b]$ is spaceable because it is an infinite dimensional vector subspace of itself and being a complete space it is closed as well.

Therefore, spaceability of a set means that the elements of the set are abundant and lineability of a set means that an extremely large subset of the set is "linear in structure."

The second author in his 1966 paper [5] proved that the set of nowhere differentiable functions is lineable. The three authors in their 2004 paper [2] prove the same result for differentiable but nowhere-monotone functions. They established the fact that the set of DNM functions is lineable. This means that there is an infinite dimensional vector subspace of functions that are everywhere differentiable and non-monotonic making such functions abundant!

Concluding Remarks:

As discussed before in this paper, the unintuitive and peculiar nature of the so called "pathological" functions led to them being named so. These functions were initially met with skepticism and disdain because they did not seem at the time of much use. They were considered as forced counter examples that lay at the fringes of the set of functions. They were dismissed as mere aberrations and were considered to be atypical. However, it was proved later that these functions were in fact not atypical at all. Specifically, the set of DNM functions is lineable which means that such functions are actually extremely abundant. In fact they are more abundant than the simple functions that we are used to, the same way irrational numbers are more abundant than rational numbers. Therefore, they cannot be considered as mere aberrations.
Thus it can be concluded that the "Pathological" functions are not really pathological the same way as "Complex" numbers are not really complex.

References

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