Showing The Surprising Difficulty of Proving That a Circle has the Smallest Perimeter for a Given Area, and Other Interesting Related Problems.

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Abstract

Throughout this paper, I shall show why a circle has the minimum perimeter for a given area, using the isoperimetric inequality, and see if this is possible for a sphere. We shall also briefly discuss an unsolved problem which relates quite well to this topic and if solved could show, in a nice way, that a sphere has the smallest ratio between surface area and volume.

1 Introduction

If I give you the following question:

> What shape or object would have to be put around a point to give the minimal perimeter for a unit area?

If you've thought about this for a short time, some of you would have possibly thought about a circle, this is usually because a teacher in high school has told us of this, but how do we know this is a fact? We may think this is a relatively simple problem to prove, but a rigorous proof was only discovered in 1841 [9], and this was only for $\mathbb{R}^2$, a proof for higher dimensions was found almost a century later in 1919 [7].

Throughout this paper, we shall explore this problem and its surprising difficulty for the $\mathbb{R}^2$ plane and then its further applications within the $\mathbb{R}^3$ plane.

2 What is the Minimum Perimeter for a Unit Area in a 2D Plane?

2.1 Conditions and Notation

To be able to understand why a circle is a very special shape, we shall look into regular n-gons and their ratios between area and perimeter. To be able to do this, we need to use the following conditions and notation:

1. All connections between nodes shall be straight lines only, which makes such a path the minimal distance between the 2 nodes.
2. All areas shall be equal to 1.
3. All nodes shall be the same distance from the origin.
4. All nodes shall have equal angles between them.
5. $||\triangle ABC||$ will denote the area of the $\triangle ABC$, and $|\overline{AB}|$ will denote the length of $\overline{AB}$.

2.2 Deriving A Formula for an n Edged Polygon

Using these rules, it would create the following 5, 6 and 7 sided shapes:

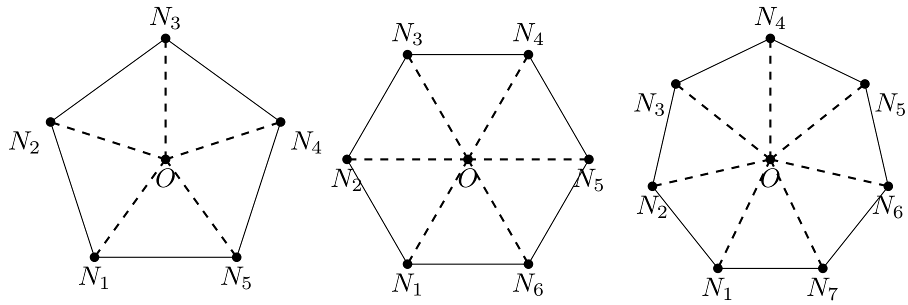

Figure 1: Depiction of 5, 6 and 7 regular sided shapes

Using Figure 1 as a guide, it is possible to derive the relationship between the area and perimeter for any given polygon; this would look something like this:

For n nodes, which shall be denoted as $\{N_1, \dots, N_n\}$. Then the following relation holds:

$$\angle N_1ON_2 = \angle N_2ON_3 = \cdots = \angle N_{n-1}ON_n = \angle N_nON_1 = \frac{2\pi}{n}. \tag{1}$$

Then the following is also true due to condition 2 and 3 above:

$$||\triangle N_1ON_2|| = ||\triangle N_1ON_2|| = \cdots = ||\triangle N_{n-1}ON_n|| = ||\triangle N_nON_1|| = \frac{1}{n}. \tag{2}$$

Then the area for any of the given triangles is:

$$||\triangle N_iON_{i+1}|| = \frac{1}{2}|\overline{N_iO}||\overline{N_{i+1}O}| \sin(\angle N_iON_{i+1}) = \frac{1}{n} \quad i \in \{3, 4, \dots, n\}. \tag{3}$$

By the following simplification and rearrangement

$$|\overline{N_iO}| = \sqrt{\frac{2}{n \times \sin(\angle N_iON_{i+1})}}. \tag{4}$$

Using the Law of Cosine and further simplification, the following is the expression for $|N_iO|$:

$$|\overline{N_iO}| = \sqrt{\frac{4(1 - \cos(\frac{2\pi}{n}))}{n \times \sin(\frac{2\pi}{n})}}. \tag{5}$$

Given this, the perimeter shall be the following:

$$Perimeter = 2\sqrt{n \tan\left(\frac{\pi}{n}\right)}. \tag{6}$$

2.3 What About an Infinite Edged Polygon?

We shall now show that the following equation has a finite answer:

$$\lim_{n\to\infty} 2\sqrt{n \tan\left(\frac{\pi}{n}\right)}. \tag{7}$$

The computation of this limit can be done when using the infinite series representation of Tangent (up to the x^2 term will be sufficient):

$$\tan(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots \tag{8}$$

We get the following:

$$\approx \lim_{n\to\infty} 2\sqrt{n(\frac{\pi}{n} + \frac{1}{3}\left(\frac{\pi}{n}\right)^3 + \dots)}. \tag{9}$$

Because $\frac{1}{3}\left(\frac{\pi}{n}\right)^3$ and all subsequent terms will be infinitely small, they shall be taken to equal zero.

$$\approx \lim_{n\to\infty} 2\sqrt{n\left(\frac{\pi}{n}\right)}. \tag{10}$$

$$= 2\sqrt{\pi}. \tag{11}$$

Using this answer, it can be shown that this results in the formula for the area of a circle:

$$2\pi r = 2\sqrt{\pi}. \tag{12}$$

$$(\pi r)^2 = \pi. \tag{13}$$

$$\pi r^2 = 1. \tag{14}$$

From this, it can be deduced that a circle has the smallest ratio between area and perimeter of any regular shape and therefore a circle has the minimal perimeter for a given area. It can be seen that the above 'proof' is really easy to obtain as anyone with an intermediate knowledge of mathematics would be able to derive this.

2.4 Introduction to the Isoperimetric Inequality

We can use the above result to understand the isoperimetric inequality which shall be used later on [1, 7]. The isoperimetric inequality is useful as it relates perimeter with area; for two dimensions it is the following:

$$4\pi A \leq P^2. \tag{15}$$

In our calculations $A = 1$ and we calculated P, it can easily be shown that the isoperimetric inequality holds:

$$4\pi \times 1 \leq (2\sqrt{\pi})^2. \tag{16}$$

$$4\pi \leq 4\pi. \tag{17}$$

It can also be seen to hold when $A = \pi r^2$ and $P = 2\pi r$.

$$4\pi \times \pi r^2 \leq (2\pi r)^2. \tag{18}$$

$$4\pi^2 r^2 \leq 4\pi^2 r^2. \tag{19}$$

Using this very useful inequality we can then define isoperimetric quotient as the following:

$$Q = \frac{4\pi A}{P^2}. \tag{20}$$

Furthermore to this, we can define the isoperimetric quotient for a regular n-gon as follows:

$$Q_n = \frac{4\pi \times 1}{(2 \times \sqrt{n \tan(\frac{\pi}{n})})^2}. \tag{21}$$

$$Q_n = \frac{4\pi}{4n \tan(\frac{\pi}{n})}. \tag{22}$$

$$Q_n = \frac{\pi}{n \tan(\frac{\pi}{n})}. \tag{23}$$

Using equation (8) and a similar limit proof, we can show that Q_n is at a minimum when $n \to \infty$, and therefore a circle has the smallest ratio between perimeter and area.

3 Proof of the Isoperimetric Inequality in $\mathbb{R}^2$

We have introduced the isoperimetric inequality in $\mathbb{R}^2$, and seen how this property shows that a circle has the smallest ratio between area and perimeter for any $2D$ shape. Even though we have introduced the inequality, we have not given a formal and rigorous proof of this inequality. Therefore, the proof of this inequality is as follows:

Theorem 3.1 (Classical Isoperimetric Inequality)**.** *For a simple closed plane curve of length L bounding an area A the classical isoperimetric inequality asserts that*

$$L^2 - 4\pi A \geq 0, \tag{24}$$

with equality only holding for a circle.

Proof. Given that arc length L can be expressed as follows

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt. \tag{25}$$

The area A enclosed by the simple closed plane can be expressed as follows

$$A = -\int_a^b y\frac{dx}{dt}\ dt. \tag{26}$$

In order to deal with the square root within the integral. Any multiple of arc length will be sufficient. The most sensible is $t = \frac{2\pi}{L}s$. Then

$$\int_0^{2\pi} \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]\ dt = \int_0^{3\pi} \left(\frac{ds}{dt}\right)^2\ dt = \frac{L^2}{2\pi}, \tag{27}$$

and then

$$L^2 - 4\pi A = 2\pi \int_0^{2\pi} \left(\frac{dx}{dt} + y\right)^2\ dt + 2\pi \int_0^{2\pi} \left[\left(\frac{dy}{dt}\right)^2 - y^2\right]\ dt. \tag{28}$$

If the following is non-negative

$$2\pi \int_0^{2\pi} \left[\left(\frac{dy}{dt} \right)^2 - y^2 \right] \, dt \geq 0. \tag{29}$$

The above inequality will be proved if we can prove the following inequality (Wirtinger's Inequality)

$$\int_0^{2\pi} \left(\frac{dy}{dt} \right)^2 \, dt \geq \int_0^{2\pi} y^2 \, dt. \tag{30}$$

Lemma 3.2. *If $y(t)$ is a smooth function with period 2π, and if $\int_0^{2\pi} y(t) \, dt = 0$, then the Wirtinger's inequality holds, with equality iff $y = a\cos t + b\sin t$.*

A proof of this lemma can be found in a book by G. H. Hardy, J. E. Littlewood, and G. Polya [5] and therefore I shall not include a proof of this lemma here. Before we apply the lemma to this proof, we will have to note that if the x-axis passes through the centre of the curve, this makes both terms, the left and the right, non-negative giving the following inequality

$$L^2 \geq 4\pi A \tag{31}$$

Given that equality holds only if

$$2\pi \int_0^{2\pi} \left(\frac{dx}{dt} + y \right)^2 \, dt = 0 \tag{32}$$

And following this, it can easily be seen that the curve must be a circle. □

4 Isoperimetric Inequality and Related Problems in $\mathbb{R}^3$

In section 2, we used regular polygons to derive a relation between the perimeter and their area, and further showed that a circle has the lowest ratio between perimeter and area of any regular polygon. Further to this, we shall explore if a similar approach will work with regular polyhedra.

Now we shall use the platonic solids, below is a table which gives the surface area for an area 1:

Polyhedron	Tetrahedron	Octahedron	Cube	Icosahedron	Dodecahedron	...	Circle
Nodes	4	6	**8**	12	**20**	...	∞
Surface Area	7.208	5.729	**6**	5.181	**5.341**	...	4.836

Table 1: The platonic solids, and their relationship between area and surface area.

From this table, we can see that, for the most part, the more nodes a polyhedra has, the smaller the surface area. By taking out the Cube and the Dodecahedron from our list, we will then get the following:

Polyhedron	Tetrahedron	Octahedron	Icosahedron
Nodes	4	6	12
Surface Area	7.208	5.729	5.181

Table 2: Relationship between area and surface area for a selected group of platonic solids.

There is something special about these polyhedra; these polyhedra have equilateral triangular faces, meaning that each of the neighbouring nodes has equal distance between them.

Another way of saying this would be that all these points are equally spaced upon the surface of a sphere and this a really interesting and unsolved problem called the 'Thomson Problem'.

4.1 Thomson Problem

This problem was posed in 1904 [4] and is a special case of a problem posed by Steve Smale [8]. The simple and formal description of this problem is the following:

> This problem is related to spherical codes, which are an arrangements of points on a sphere such that the minimum distance between any pair of points is maximised.

The mathematics of this problem is the following:

$$U_{ij}(N) = k_e \frac{e_i e_j}{r_{ij}} \tag{33}$$

Where k_e is Coulomb's constant [2] and $r_{ij} = |r_i - r_j|$, where r_i and r_j are defined by vectors, respectively. Using simplified units (because units will not change the solutions of the problem), we shall take $e = 1$ and $k_e = 1$. Then,

$$U_{ij}(N) = \frac{1}{r_{ij}}. \tag{34}$$

Then the total potential energy for the N-electron, can be expressed as the following,

$$U_{ij}(N) = \sum_{i<j} \frac{1}{r_{ij}}. \tag{35}$$

There are only a few known configurations which have been rigorously proved. These are the following:

- For $N = 1, 2, 3$, the solutions are quite trivial. $N = 1$ lies anywhere on the sphere, $N = 2$ lies at diametrically opposite, and $N = 3$ lies on a great circle. It can be seen that each of these solutions are 2-dimensional.
- For $N = 4, 6, 12$, are the platonic solids; tetrahedron, octahedron and icosahedron, respectively.

There are also many numerically close approximations to Thomson's problem for unsolved N, which can be used to produce a uniform triangular mesh upon the sphere. By doing this, it would allow us to produce an approximation to a regular N-node triangular-faced polyhedron, but can also be called Geodesic polyhedron.

This is useful in the context of the minimum ratio of surface area and volume because it is now possible to get a close approximation for a surface area for N-nodes. This would give an asymptotic behaviour to the ratio of that of a sphere, which can be seen in table 1.

This cannot be taken as a rigorous proof of this, but more of an idea and way of convincing yourself that a sphere has in-fact the smallest ratio between surface area and volume of any $\mathbb{R}^3$ object.

4.2 The Isoperimetric Inequality in $\mathbb{R}^3$

The previous section explored a way of thinking and loosely explained why a sphere has the smallest ratio between surface area and volume. This section will highlight how this problem is more complex in the $\mathbb{R}^3$.

This inequality is,

$$36\pi \times V^2 \leq A^3 \tag{36}$$

Where V is volume and A is surface area. It can easily be checked that this does hold as an equality for a sphere,

$$36\pi \times (\frac{4}{3}\pi r^3)^2 \leq (4\pi r^2)^3 \tag{37}$$

$$36\pi \times \frac{16}{9}\pi^2 r^6 \leq 64\pi^3 r^6 \tag{38}$$

$$64\pi^3 r^3 = 64\pi^3 r^6 \tag{39}$$

This inequality can also be shown to holds for a cube,

$$36\pi(a^3)^2 \leq (6a^2)^3 \qquad \text{Where } a \text{ is a length of an edge.} \tag{40}$$

$$36\pi a^6 \leq 216a^6 \tag{41}$$

$$36 \times 3.142a^6 \leq 216a^6 \tag{42}$$

$$113.1a^6 \leq 216a^6 \tag{43}$$

It can be seen that this also holds for a cube, and this would be expected.

The inclusion of the proof of the spherical isoperimetric inequality here would not benefit the paper and would be incomprehensible for most readers. This, unfortunately, includes trying to provide any examples of this, as the mathematics is quite nuanced and would be quite difficult to understand. Rigorous proofs of both of these inequalities can be found in the relevant citations [7, 1].

4.3 Kelvin Problem

This is a very interesting problem that Lord Kelvin asked; 'how could space be partitioned into cells of equal volume that has the least surface area between them' [3]. This problem is the $\mathbb{R}^3$ variant of the Honeycomb conjecture, which was proven in 1999 [6].

Interestingly, the current solution for the Kelvin Problem was found in the 1993 and this object is called the Weaire-Phelan structure. This structure is a species of tetrakaidecahedron (which just means an object with 14 sides)

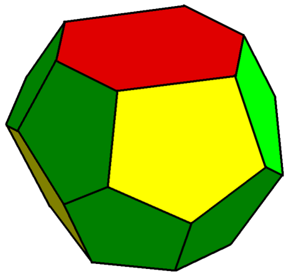

Figure 2: A tetrakaidecahedron

This solution has not been proven to be the best shape for this problem and in 2009 a new computer-aided search was started to try and find further object/objects which could improve on the current solution. This is also an important problem as it relates well to the manufacturing of bubble wrap, as manufactures are always wanting to reduce the amount of materials used for the same volume of bubble wrap.

5 Conclusion

This paper explored a very simple question that has more complicated solutions that would be expected. These solutions were known many many centuries ago, and it took mathematics which was developed within the last 100 years to be able rigorously solved. We have further looked into if the isoperimetric inequality also works in $\mathbb{R}^n$, even though the proof would be too complicated to discuss. This is odd because it wouldn't be expected such a problem would have such a complicated solution.

Further discussed in this paper is other unsolved problems which are simple at their core, but are yet to be solved. This is interesting as solutions to these simple problems could have some very interesting applications within science and manufacturing as a whole.

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