The Fibonacci Numbers and The Golden Number

Andrea Pygott

12th April 2019

1 Abstract

In this paper, I will be exploring the Fibonacci numbers and their applications in the natural world. I will be discussing how the Fibonacci sequence also relates to the golden number, and then how both of these can be found in nature.

2 Introduction

People who have studied maths to a certain level will have heard of Fibonacci and his numbers, and possibly even the golden number. What many people do not understand is their application, both in mathematics and the real world. The Fibonacci sequence is also closely related to something called the golden number, and we will see how later in the paper. In this paper, we will be exploring what the Fibonacci sequence is, where it came from and where it appears in nature and history, and how the sequence links to the golden number and prime numbers in mathematics.

3 What is the Fibonacci sequence?

Firstly, we will try to understand the Fibonacci numbers and the sequence they form. The Fibonacci sequence is credited to an Italian mathematician called Leonardo Pisano[15] (or Leonardo of Pisa), however, he did not discover the sequence himself. In fact, the sequence was known much earlier - possibly as early as the 6th century AD - by Indian mathematicians[15]. But, Leonardo introduced the sequence of numbers to the western world after travelling across the Mediterranean and Northern Africa [15]. Fibonacci numbers are usually denoted by F_n, and they form a sequence such that every number in the sequence is the sum of the two numbers that come prior to that particular number, and it starts with 0 and 1, such that:

$$F_0 = 0, F_1 = 1 \tag{1}$$

The general equation for the sequence is given by

$$F_n = F_{n-1} + F_{n-2} \tag{2}$$

and n has to be greater than 1 (n>1). The sequence of Fibonacci numbers then looks like this

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \ldots \tag{3}$$

We don't often look at the terms of the Fibonacci sequence below zero, but they can be calculated. In fact, the terms below zero are relatively the same as the terms above zero, however, they follow a + - + pattern. It goes like so

$$\ldots, -377, 233, -144, 89, -55, 34, -21, 13, -8, 5, -3, 2, -1, 1, 0, \ldots \tag{4}$$

The terms below zero can be given by a different formula for their sequence, and this is given by

$$F_{-n} = (-1)^{n+1} F_n \tag{5}$$

But this is basically working backwards from zero, so we can also write this as

$$F_n = F_{n-2} - F_{n-1} \tag{6}$$

But, because we are working backwards from zero, that does not make the terms below zero a different sequence to the terms above zero. It depends where we start in the sequence. For example,

$$-377 + 233 = -144 \tag{7}$$

which is the same sequence as before, we have just started below zero. In fact, we can start with any numbers we want. Say, we start with the numbers 1 and 3. The next term would be 4, and the next 7 and so on. This particular sequence has its own name and that is the Lucas sequence [13] (if you want to learn more about this sequence, you can follow the url in the reference section). However, the Fibonacci sequence is a specific sequence that starts with 0 and 1, and this is the specific sequence we will be looking at in this paper.

There are also many formulae to find the n^{th} Fibonacci number in the sequence, however, we will be looking at Binet's formula to do this. The formula is given as follows [11]:

$$F_n = \frac{1}{\sqrt{5}}\left[\left[\frac{1+\sqrt{5}}{2}\right]^n - \left[\frac{1-\sqrt{5}}{2}\right]^n\right] \tag{8}$$

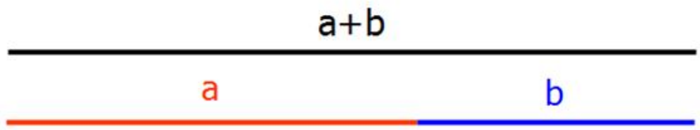
So, to give an example of how to use this formula, say we want to find the 22^{nd} Fibonacci number. Then we would use $n=22$ and sub 22 into the formula given above like so

$$F_{22} = \frac{1}{\sqrt{5}}\left[\left[\frac{1+\sqrt{5}}{2}\right]^{22} - \left[\frac{1-\sqrt{5}}{2}\right]^{22}\right] \tag{9}$$

Once we've simplified this equation, we get $F_{22} = 17711$, and this can be checked, this is correct. So, we have a way of finding the n^{th} Fibonacci number without needing to know the $n^{th} - 1$ and $n^{th} - 2$ Fibonacci numbers.

4 The Golden Number

The Fibonacci sequence has a close relationship to the golden number. You may be wondering what the golden number is, and why it is important. The golden number is a ratio, and it has been labelled phi - a Greek letter - and this is denoted by φ. The golden number is approximately equal to 1.618 [15]. There are a few ways in which we can find the golden ratio. The first way is to take a line on a 2D plane, we then split the line into two, such that one segment is longer than the other. We find the golden ratio by doing the following calculations with Figure 1.



[8]

Figure 1

$$\frac{a+b}{a} = \frac{a}{b} = \varphi = 1.618\ldots \tag{10}$$

The golden number is found when the two segments of the given line are split in a certain ratio, and depending on how long the original line is, gives the length of the two segments it is divided into. As we can see from equation (10), the fraction $\frac{a+b}{a}$ is the same as $\frac{a}{b}$, and the segments of the line - a and b - have to be such that this fraction is equal to φ

The golden number is irrational, just like pi (π). And also just like pi, the decimal numbers are random and they carry on infinitely. The golden number can also be expressed in terms of itself, like so

$$\varphi = 1 + \frac{1}{\varphi} = 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\ldots}}}} \tag{11}$$

[8] The fraction above in (11) is called a continued fraction, and this is because it just carries on and on. How does the golden number relate to the Fibonacci sequence? Well, we can use the Fibonacci sequence to approximate the golden number. If we take the Fibonacci sequence to be as it was before:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots \tag{12}$$

then the ratio between two consecutive numbers in this sequence is a very close approximation to the golden number. For example, if we take the numbers 5 and 8 - they are in the Fibonacci sequence. We then divide 8 by 5 to get the ratio between them and this is equal to 1.6. You can see this in the following equation

$$\frac{8}{5} = 1.6 \tag{13}$$

In fact, we can generalise this equation into a formula for the terms in the Fibonacci sequence for when n is the nth term in the Fibonacci sequence and n is greater than 0. This generalisation does not work when n=0 because $n_0 = 0$ and for this calculation (for the first two terms, 0 and 1) we would be dividing 1 by 0 and it is impossible to divide a number by zero.

$$\frac{F_n}{F_{n-1}} \approx 1.6 \approx \varphi \tag{14}$$

We can also see this from the table provided below by Figure 2.

First Number	Second Number	Ratio between the numbers
1	1	1
1	2	2
2	3	1.5
3	5	1.666666667
5	8	1.6
8	13	1.625
13	21	1.615384615
21	34	1.619047619
34	55	1.617647059
55	89	1.618181818
89	144	1.617977528
144	233	1.618055556
233	377	1.618025751
377	610	1.618037135
610	987	1.618032787
...	...	...
46368	75025	1.618033989
75025	121393	1.618033989
121393	196418	1.618033989
196418	317811	1.618033989
...	...	...

Figure 2

As we can see from the table above the larger the two consecutive numbers we take, the closer the approximation is to the real value of φ. Since we used 8 and 5 earlier, the approximation for φ from these two numbers is 1.6. However, if we take 317811 and 196418 then we get an approximation of 1.618033989. The actual value of the golden number is 1.61803398874989484820... to 20 decimal places [8].

5 Mathematics and History

The Fibonacci numbers have a lot of history behind themselves, however, in this section, we will be exploring how the Fibonacci numbers have been used in mathematics and their applications in the history of mathematics. We will specifically look at the application of the Fibonacci numbers with prime numbers and divisibility and how the Greeks used the Fibonacci numbers alongside the golden number in art and architecture.

5.1 Primes and Divisibility

Many people do not know this, but the Fibonacci sequence links quite well with prime numbers and the sequence is actually a divisibility sequence.

If we take a look at the Fibonacci sequence we find something interesting. If we specifically analyse the last digit of each of the numbers,

$$\mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{5}, \mathbf{8}, 1\mathbf{3}, 2\mathbf{1}, 3\mathbf{4}, 5\mathbf{5}, 8\mathbf{9}, 14\mathbf{4}, 23\mathbf{3}, 37\mathbf{7}, 61\mathbf{0}, \ldots \tag{15}$$

then we notice that these numbers keep repeating over and over forever - however, it is after the first 60 numbers in the sequence, so it takes a while to notice[14]. We can also do this with the last two digits of the Fibonacci numbers, they also repeat over and over again forever. However, this time the cycle length is 300. This means that the last two digits repeat themselves, but only after you reach the 300th Fibonacci number. In fact, we can do this for a higher number of digits as well. The cycle length for the last three digits is 1,500, the cycle length for the last four digits is 15,000, and for the last five digits it is 150,000, and so on [14].

If we decide to analyse the Fibonacci sequence as a whole, then we can see that every third number in the sequence is even.

$$\mathbf{0}, 1, 1, \mathbf{2}, 3, 5, \mathbf{8}, 13, 21, \mathbf{34}, 55, 89, \mathbf{144}, 233, 377, \mathbf{610}, \ldots \tag{16}$$

In fact, we can say something a little further, which is that the Fibonacci sequence satisfies the stronger divisibility property such that

$$gcd(F_m, F_n) = F_{gcd(m,n)} \tag{17}$$

[5] This means that the greatest common divisor of two Fibonacci numbers is equal to the greatest common divisor of the two specific term's Fibonacci number. These two Fibonacci numbers have to be consecutive in the sequence for this rule to work. An example may be a better way of explaining this, so we will go through an example.

Suppose we have the Fibonacci numbers $F_m = F_8 = 21$ and $F_n = F_9 = 34$ - so *m=8* and *n=9*. Then the $gcd(F_8, F_9) = gcd(21, 34) = 1$ and the *gcd(8,9)=1* and $F_{gcd(8,9)} = F_1 = 1$. So, this property is satisfied.

We will now move onto Fibonacci numbers and their factors (and prime factors).

There is a surprising fact that generally, not many people know, and this is that there are infinitely many Fibonacci numbers with any given integer is a factor [14]. This means, you could pick any integer and it will be a factor of at least one Fibonacci number - so it divides at least one Fibonacci number. This can be shown from the table below

Integers	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
Fibonacci number	1	2	3	8	5	144	21	8	144	610	55	144	13	46368	6765	...
N-th term in sequence	1	3	4	6	5	12	15	6	12	15	10	12	7	24	20	...

[14]

Figure 3

Another aspect of Fibonacci numbers that is quite interesting is that if the n^{th} term is a prime number, then the Fibonacci number is also prime. This means that if the subscript n in F_n is a prime number, then the number F_n itself is itself prime. An example of this is the 7th term in the sequence, $F_7 = 13$. 7 is a prime, and so therefore 13 must also be prime, and this is true. This is true for all Fibonacci numbers with a prime subscript [14]. Proof of this is easy enough to find. In fact, you can find a list of prime Fibonacci numbers that have a prime subscript, however, the list will forever be incomplete because there are infinitely many prime numbers - so there will be infinitely many prime Fibonacci numbers - and some of the Fibonacci numbers that should be prime are yet to be proven so. Of course, so far the rule holds, but if it ever found to be untrue for one term, then we cannot guarantee the rule to then be true for all terms. We may find that it is only true for the first however many prime subscripts.

5.2 How the Ancient Greeks used the Golden Number

A perfect example of the use of the golden number in history is that of the construction of a building in Ancient Greece. The Acropolis is an outcrop of rock above the city of Athens in Greece. One of the most famous monuments is the Parthenon, which is a temple dedicated to the Goddess Athena, and it was built around the 5th century BC [12]. There are no original plans of the Parthenon, however, it appears to be built by a design that revolves around golden rectangles. A golden rectangle is a rectangle whose sides are in the ratio of the golden number, like so $1 : \varphi$ (or $1 : \frac{1+\sqrt{5}}{2}$ [3]. This means that if the rectangle has two sides of length a and two sides of length b, then the ratio of length a to b is φ. So, for example, if the longer sides are a and the shorter sides are b then these two sides are in the ratio $1 : \varphi$. If we add a square with sides of length a next to a golden rectangle with the sides a and b, then we achieve another golden rectangle with the sides of length $a+b$ and b [12].

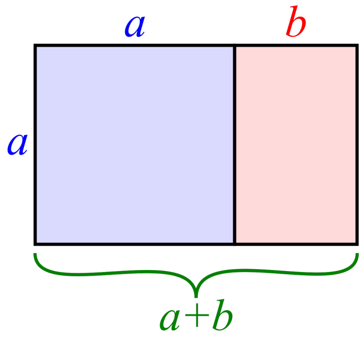


Figure 4

As we suggested earlier, there are no original plans of the Parthenon, however, we know that it's built with rectangles with lengths of the golden number and $\sqrt{5}$. Since there are no original plans left, we cannot ascertain whether the Ancient Greeks built the Parthenon specifically with the sides of the rectangles in the ration of the golden number. However, we have a good idea about why this may have come about, and that is because the building is aesthetically pleasing - meaning it looks good on the eye. The Ancient Greek society was incredibly advanced in the ways of mathematics and sciences, so it is perfectly plausible that they did build their buildings, temples and monuments around how the architecture fitted together mathematically, but it is also possible that they did not figure out the mathematics behind the aesthetically pleasing plans they created. And we cannot yet determine which it is - maybe it is a bit of both aspects.

From the figure below, you can see an image of a Greek temple (the Parthenon), and next to it the mathematical relationship drawn on two diagrams [12][1][2].

Figure 5

From Figure 5 we can see the relationship between the Fibonacci spiral and the golden rectangle and the Parthenon. This illustrates the great work of the architects that built this magnificent building. We can see that each rectangle within the golden rectangle in the diagram is itself a golden rectangle, and we can see how the Fibonacci spiral (see more about this spiral in section 6.2) connects to the golden rectangles and the Parthenon itself.

6 Fibonacci in Nature

Surprisingly, the Fibonacci sequence can be found in many places that people do not even think of or know about. Examples of uses of the Fibonacci sequence or numbers in mathematics are: they are important in the computational run-time analysis of Euclid's algorithm to determine the greatest common divisor of two integers, Yuri Matiyasevich showed that the Fibonacci numbers can be represented by a Diophantine equation and this led him to solve Hilbert's tenth problem, they can be used in planning poker, number generators, they are found in the analysis of Fibonacci heap data structure[5]. Another great use for the Fibonacci numbers is converting miles to kilometres. So suppose we take 13 and 21 as the two Fibonacci numbers, then the smaller number - 13 in this case - is the number of miles to the larger number - 21 in this case - of kilometres. (13 miles is equivalent to 20.92 kilometres, so this method is no exact, but for everyday use, it works well.) The Fibonacci numbers and the sequence they form can also be found in aspects of nature.

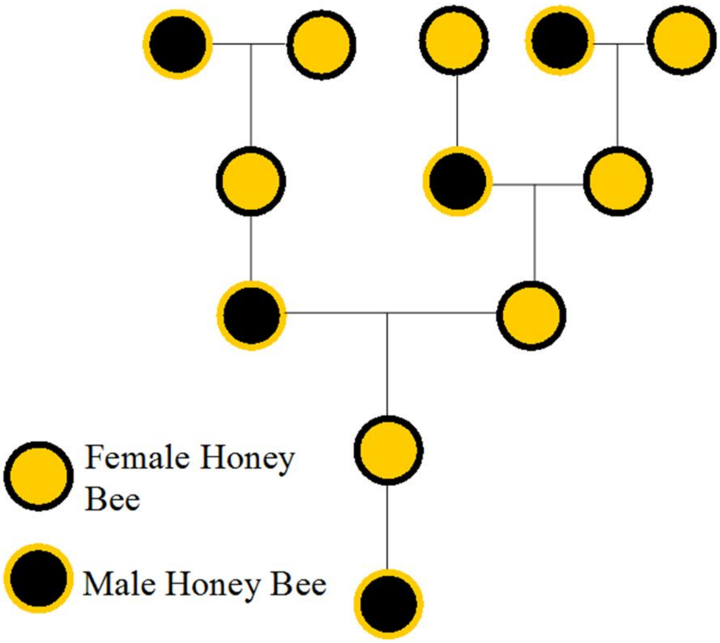
> "If a twig or branch is selected, and, starting with one bud, one's hand moves to the nearest bud and so continues in spiral fashion around the twig until it reaches a bud just above the starting one, the number of intervening buds will be one of the Fibonacci numbers, and the number of revolutions around the stalk will be another such number, which ones depending on the particular plant." [10]

You can see from this quote that branches and buds on trees and plants are arranged in the Fibonacci sequence.

Other examples in nature include things such as the arrangements of leaves on a stem, the fruitlets of a pineapple, the flowering of an artichoke, the arrangement of a pine cone and the family tree of honey bees[5].

6.1 Honey Bees

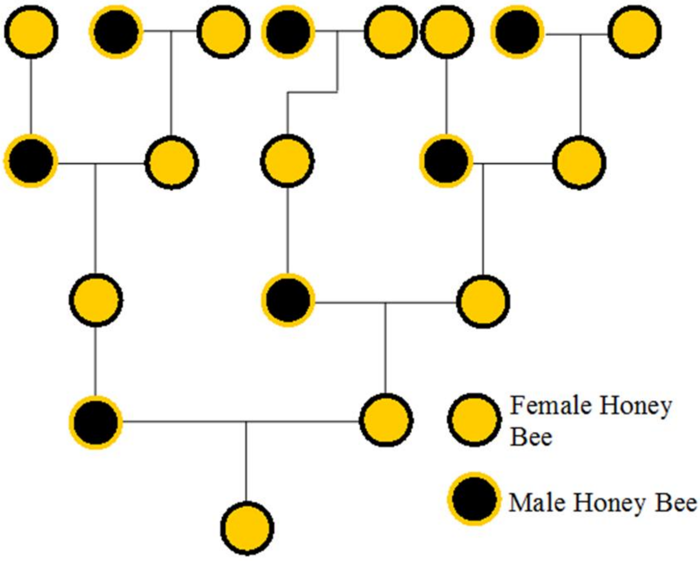
In this small section, we will explore how the Fibonacci sequence relates to honey bee's family trees. Many people may not know that not all honey bees have two parents [6]. In fact, female honey bees are born when the Queen mates with another male (honey) bee; and the male bees are born from the unfertilised Queen's eggs. This means that male honey bees only have one parent. Now, let us analyse the male honey bee's family tree in Figure 6.



[6]

Figure 6

As we can see from the family tree, starting from the 'youngest' male bee at the bottom, he has one parent, then 2 grandparents, and 3 great grandparents and so on. This forms the Fibonacci sequence starting from 1 like so: 1,1,2,3,.... Similarly, we can look at the family tree for a female honey bee and get a similar result. Take a look at Figure 7 below.



[6]

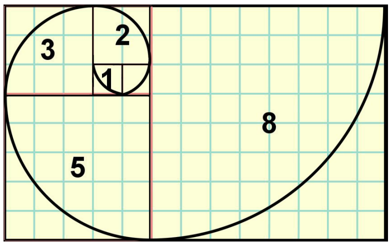
Figure 7

From the family tree of a female honey bee, she has 2 parents, then 3 grandparents, and then 5 great grandparents and so on. Again, this forms the Fibonacci sequence, but it starts in a different place - we start from the second 1 instead of the first 1, like so: 1,2,3,5,.... And so, hopefully, one has learnt about how honey bees and their family trees are related to the Fibonacci sequence.

6.2 Snail Shells

Spirals can be found in architecture and in nature. We have spiral staircases for decoration, but also spirals can be found on the backs of snails. The shell of a snail is spiral shaped because of the way snails reproduce and grow - and their shells can be different in size, shape and colour depending on the species of the snail. However, we are not here to be discussing the biology of the snail, but the mathematics of the spiral on it's back. Most mathematicians will know that the spiral is closely related to the Fibonacci sequence. Firstly, we start by drawing a square. Suppose this square has lengths of 1 unit. We then copy this square, with lengths of 1 unit, and place it next to the original square. After this, we can make another square using the longer side of the rectangle we've made from the two smaller squares, and this new square's side will be of length 2 units. Are you noticing a pattern? We carry on with this process and get squares with lengths matching the Fibonacci numbers, and they are already starting to spiral around each other. To make the spiral itself, we join the opposing angles in the squares with a curve. This whole process can be seen in Figure 8 below.

Figure 8



[16]

This particular spiral is called a Fibonacci spiral since it uses the Fibonacci numbers to be constructed. As we have seen earlier in this paper, the Fibonacci sequence can be used as an approximation tool to find the golden number. When the Fibonacci numbers get closer to infinity, the spiral gets closer and closer to approximating the golden number too when more and more squares are added [4].

7 Conclusion

To conclude this paper, we have seen how the Fibonacci sequence is formed by starting with 0 and 1 and then adding the previous two terms together to get the next number in the sequence. We have seen that the golden number φ can be approximated using the Fibonacci sequence, how to find any given Fibonacci number and how the sequence is linked with prime numbers in mathematics. Finally, we have also seen where the Fibonacci numbers can be found in nature - in the family tree of honey bees and the spirals of snail shells.

References

[1] 2019. URL: https://www.google.com/search?q=parthenon+golden+rectangle&hl=en-GB&authuser=0&source=lnms&tbm=isch&sa=X&sqi=2&ved=0ahUKEwiM_9nL3LPhAhUWKLkGHdU5DXoQ_AUIDigB&biw=1366&bih=625#imgrc=MaODCsicgob5sM:.

[2] 2019. URL: https://www.google.com/search?q=parthenon+golden+rectangle&hl=en-GB&authuser=0&source=lnms&tbm=isch&sa=X&sqi=2&ved=0ahUKEwiM_9nL3LPhAhUWKLkGHdU5DXoQ_AUIDigB&biw=1366&bih=625#imgrc=_uSlC5Odiy1M3M:.

[3] Various Authors. *Golden Rectangle*. 2019. URL: https://en.wikipedia.org/wiki/Golden_rectangle.

[4] Various Authors. *Golden Spiral*. 12th March 2019. URL: https://en.wikipedia.org/wiki/Golden_spiral.

[5] Various authors. *Fibonacci number*. 12th March 2019. URL: https://en.wikipedia.org/wiki/Fibonacci_number.

[6] Dothefinancial Blog. *Honeybees Fibonacci numbers and Family trees*. 13 November 2018. URL: https://www.dothefinancial.info/fibonacci-numbers/honeybees-fibonacci-numbers-and-family-trees.html.

[7] Maths is Fun. *Fibonacci Sequence*. 8th March 2019. URL: https://www.mathsisfun.com/numbers/fibonacci-sequence.html.

[8] Maths is Fun. *Golden Ratio*. 8th March 2019. URL: https://www.mathsisfun.com/numbers/golden-ratio.html.

[9] Maths is Fun. *Nature, The Golden Ratio, and Fibonacci too...* 12th March 2019. URL: https://www.mathsisfun.com/numbers/nature-golden-ratio-fibonacci.html.

[10] Burton W Jones. *The Theory of Numbers*. 1955.

[11] James P Jones. *Diophantine Representation of the Fibonacci Numbers*. URL: https://www.fq.math.ca/Scanned/13-1/jones.pdf.

[12] Dr Ron Knott. *Fibonacci Numbers and The Golden Section in Art, Architecture and Music*. 22 September 2010. URL: http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibInArt.html.

[13] Dr Ron Knott. *The Lucas Numbers*. 23 Januray 2017. URL: http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/lucasNbs.html.

[14] Dr Ron Knott. *The Mathematical Magic of the Fibonacci Numbers*. 30 August 2016. URL: http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibmaths.html.

[15] Gary Meisner. *What is the Fibonacci Sequence?* 15th May 2012. URL: https://www.goldennumber.net/fibonacci-series/.

[16] Akash Peshin. *What is the Fibonacci Sequence? Why is it so special?* 2018. URL: https://www.scienceabc.com/eyeopeners/why-are-fibonacci-numbers-so-important.html.

[17] BioExpedition Publishing. *Snail Anatomy*. 2017. URL: https://www.snail-world.com/snail-anatomy/.