A Mathematical Deconstruction of Pyramid Schemes

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Abstract

Pyramid schemes are often seen as easy ways to make money. Within this paper mathematics is used to challenge this notion. It aims to show how misleading and harmful the seemingly inoffensive business model is, by critiquing the points of failure built into its structure.

1 Introduction

A pyramid scheme is a business model that gains members or customers by promising financial returns on an initial investment by enrolling or recruiting more members into the scheme. Pyramid schemes are similar to middle-level management businesses[4]. However, middle-level management schemes focus on selling a product rather than recruiting more members to the business[3]. Pyramid schemes are named as such since at each level down from the top, there must be more members than in the previous level for the model to work, resulting in a pyramid shape of members. Most pyramid schemes are illegal because they are unsustainable and do not offer a reward equal to the value of any sign-up fee. Many of these schemes come under different guises, such as chain letters. Within this paper, I will be looking at different types of pyramid schemes and using mathematics to understand how they work or do not work in practice.

2 The Model

Pyramid schemes start with an individual or company (level 1 of the scheme) reaching out to a number of other people, and asking for a sum of money, in return for involvement in the 2nd level of the scheme and a promise of their portion of the profits. Then the recently recruited level 2 members recruit their own members and receive payment from the next level and so on. For example, the simple model pyramid scheme could work by each member paying a sign-up fee of £100 to the person that brought them to the system. Then once in the pyramid, they can recruit their own members and keep any money that they get from other people signing up. Let's imagine everyone in the scheme recruits two members below them in the pyramid, then everyone stands to double their money to £200. Therefore, these schemes can be so popular because it seems at a glance that everyone can win.

Another version of the pyramid scheme is the 8-Ball model, the principle is the same however, a member receives their profit from the 3rd level below the one that they are on. For example, someone at level three might pay £100 to enter the scheme and recruit two people in level four. These then recruit 4 people in level five, who in turn recruit 8 people in level six themselves. Then the person in level three would receive the eight payments of £100 recruited below them in level 6. This holds an emphasis on members helping others in the lower levels as their success aids their own. In this model, each member is promised eight times the money they invested if each member recruits two more members. Once again it seems like there is easy money to make by entering these schemes, but this is not the case.

3 Exponential Growth

Regardless of which model is used the structure of the scheme is defined by each level having more members than the last. Whether each member recruits two, three, four or more members, the number at the next level is proportional to the number of members at the current level. This kind of growth is called exponential growth and is seen in other areas of the world, such as the multiplication of bacteria and compound interest. If we assume that each member recruits two more then each level will be twice as big as the last level and each level will contain 2^{N-1} members, N being the number of the level. If each member recruits M members below them then each level will contain M^{N-1} members. The table below shows the total number of members in a pyramid based on the number of levels (N) and the number of people each member has recruited (M). Each cell entry consists of the sum, for all levels of the pyramid, of the number of members in that level of the pyramid. As at any level, M^{N-1} is the number of members at level N of a pyramid that recruits M members each. This formula is shown here:

$$\text{Cell entries} = \sum_{N=1}^{N} M^{N-1} \tag{1}$$

	M = 2	M = 3	M = 6	M = 8
N = 1	1	1	1	1
N = 2	3	4	7	9
N = 3	7	13	43	73
N = 4	15	40	259	585
N = 5	31	121	1,555	4,681
N = 6	63	364	9,331	37,449
N = 7	127	1,093	55,987	299,593
N = 8	255	3,280	335,923	2,396,745

This shows that at level 8 the growth for a scheme with M = 8 already has over 2 million members. Furthermore, we can calculate for which N, each pyramid of level growth M, will outgrow the population of the earth, meaning we know the maximum amount of levels each type of pyramid could possibly contain.

3.1 Case M=2

We want to find a minimum N, such that:

$$\sum_{N=1}^{N} 2^{N-1} > 7{,}684{,}000{,}000 \text{[1]} \tag{2}$$

We are taking the current population as of February 2019 as 7.684 Billion as cited. We can re-write the left-hand side of this inequality to be

$$\sum_{N=1}^{N} 2^{N-1} = 2^N - 1 \tag{3}$$

So we now need to find a minimum N, such that

$$2^N - 1 > 7{,}684{,}000{,}000 \tag{4}$$

Adding 1 to both sides leaves us with

$$2^N > 7{,}684{,}000{,}001 \tag{5}$$

Using the definition of a logarithm we can change this to

$$N > log_2(7{,}684{,}000{,}001) \tag{6}$$

We can re-write the right-hand side of this equation to more easily understand it

$$N > \frac{\ln(7{,}684{,}000{,}001)}{\ln(2)} \tag{7}$$

If we calculate the right-hand side of this equation, we can evaluate a minimum N

$$\frac{\ln(7{,}684{,}000{,}001)}{\ln(2)} = 32.839 \tag{8}$$

So the number of levels in the pyramid that will cause the size of the whole pyramid to exceed the population of the earth is 33 if M = 2. Therefore, theoretically this pyramid can withhold 32 full levels of members.

3.2 Case M

From this calculation, we can adapt the equation for any pyramid and any M to calculate at what level any given pyramid scheme would exceed the population constraint.

Let P be the current human population of the earth.

Let M be the number of new recruits each existing member in the pyramid scheme enlists.

Then the number of the first level that will tip the number of total members of the scheme over the human population is

$$N \in \mathbb{Z},\ such\ that,\ \lfloor N \rfloor \geq \frac{\ln(P+1)}{\ln(M)} \tag{9}$$

As we saw, for M = 2 this N is 33.

$$\text{For M = 3,} \qquad \lfloor N \rfloor \geq \frac{\ln(7{,}684{,}000{,}001)}{\ln(3)}, \text{ we get N = 21} \tag{10}$$

$$\text{For M = 6,} \qquad \lfloor N \rfloor \geq \frac{\ln(7{,}684{,}000{,}001)}{\ln(6)}, \text{ we get N = 13} \tag{11}$$

$$\text{For M = 8,} \qquad \lfloor N \rfloor \geq \frac{\ln(7{,}684{,}000{,}001)}{\ln(8)}, \text{ we get N = 11} \tag{12}$$

If there are no members to recruit to a pyramid scheme then the members in the bottom level have no reason to remain in the scheme and leave, this effect then ricochets up the pyramid and the whole scheme collapses quickly. This shows that these schemes are not sustainable in the long term.

4 Who makes the money?

We have observed that there can be many people involved in a pyramid scheme. We can also calculate the amount of money involved in any such scheme as we know at any point how many members there are.

Let N be the number of levels in the scheme.

Let M be the number of new recruits each existing member in the pyramid scheme enlists.

Let T be the total money in the scheme.

Let S be the sign-up fee for all members.

Then from our previous calculations

$$T = S \times (M^N - 1) \tag{13}$$

This is a simple calculation but we can intuitively see that the amount of money in a scheme will grow proportionally with the number of members. If the number of members grows exponentially and the amount of money grows proportionally to the number of members then, so too, will the amount of money.

Now we will look at who all this money goes to in the simple and the 8-ball model. For the simple model, each member gets paid as soon as they recruit the members beneath them, so only the final level of members are down money at any given time. For the simple model where M = 2 we know that the proportion of members who are currently losing money is the number of members in the lowest level of the pyramid divided by the total members in the scheme.

$$\frac{M^{N-1}}{M^N - 1} \tag{14}$$

4.1 Simple model, Case M = 2

If we apply this to the case M = 2

$$\frac{2^{N-1}}{2^N - 1} \tag{15}$$

We can re-write the numerator of this fraction

$$2^{N-1} = 2^N \times 2^{-1} = \frac{1}{2} \times (2^N) \tag{16}$$

So we have

$$\frac{\frac{1}{2} \times (2^N)}{2^N - 1} \tag{17}$$

If we add one to the denominator, then we know that our fraction will be larger than this

$$\frac{\frac{1}{2} \times (2^N)}{2^N - 1} > \frac{\frac{1}{2} \times (2^N)}{2^N} \tag{18}$$

We cancel the equivalent expressions in the fraction

$$\frac{\frac{1}{2}\times(2^N)}{2^N-1} > \frac{1}{2} \tag{19}$$

$$\frac{2^{N-1}}{2^N-1} > \frac{1}{2} \tag{20}$$

So at any given time, and any given levels in the pyramid, more than half the members of the simple pyramid scheme with M = 2 have lost money in the scheme. Although there can be a lot of money in these schemes, less than half of the members can actually profit.

4.2 Simple model, Case M

Instead of M = 2, lets replace 2 with M in our previous equation to get the general case:

$$\frac{M^{N-1}}{M^N-1} > \frac{1}{M} \tag{21}$$

This means that the more people each member recruits, the smaller the fraction of people in the scheme that are losing money can be.

4.3 8-Ball model

In the 8-Ball model the big difference compared to the simple one is that the bottom three levels of members are not earning money. This is as a result of each member only receiving money from the level that is three below them, so levels N, N-1 and N-2 do not have these levels to profit from yet. Our new equation for the proportion of members losing money in an 8-Ball model will be the number of members in the bottom three levels, divided by the total number of members in the model.

$$\frac{2^{N-1}+2^{N-2}+2^{N-3}}{2^N-1} \quad [2] \tag{22}$$

If we add one to the denominator, then we know that our fraction will be larger than this

$$\frac{2^{N-1}+2^{N-2}+2^{N-3}}{2^N-1} > \frac{2^{N-1}+2^{N-2}+2^{N-3}}{2^N} \tag{23}$$

We can factorise out 2^{N-3} from the numerator and denominator

$$\frac{2^{N-1} + 2^{N-2} + 2^{N-3}}{2^N - 1} > \frac{2^{N-3}\,(2^2 + 2^1 + 2^0)}{2^{N-3}\,(2^3)} \tag{24}$$

If we cancel the equivalent expressions and simplify then

$$\frac{2^{N-1} + 2^{N-2} + 2^{N-3}}{2^N - 1} > \frac{7}{8} \tag{25}$$

The 8-Ball model in which it seems as though you can make back eight times your money, actually only allows less than one-eighth of its members to profit.

5 Conclusion

As we have seen, pyramid schemes cannot last, as the human population limit means that eventually there will be no one left to recruit and the scheme will collapse. When this happens, in the simple model, over half of the members will have lost money. This seems as though, in the big picture, the bottom “half” of the pyramid have paid the top “half” of the pyramid for nothing. In the 8-Ball model, however, less than one-eighth of the members have octupled their money, directly from over seven-eighths of members losing theirs. This is a clear indication of why these schemes are largely illegal and can cost people a lot of money.

6 References

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