Mathematics of Campanology

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1 Abstract

In this paper I will be exploring the ways in which mathematics is used in the hobby of campanology. I will be using concepts of algebra such as cycles and permutations to explain 'methods' - the patterns the bells make when ringing.

2 Introduction

Many people across the world will have heard the sound of a church bell. They have been present throughout history at important events such as royal weddings, coronations, funerals and, most recently, the one-hundredth year anniversary of the end of The Great War. For those who haven't heard the sound of church bells ringing, a quick search on the internet will give a good idea as to how they sound. Many people have heard of Big Ben in London, England. It might be quite hard to imagine that behind bell ringing there is some core mathematics, and many campanologists do not even realise this themselves, but in this paper, I am going to explain the mathematics that is included in campanology.

3 Numbers and Bells

You will not be surprised that numbers are included in mathematics since they are the backbone of everything that is mathematics. However, you may be surprised that numbers are also very much a part of bell ringing. For the moment, we will forget about the bells and concentrate on the numbers. Suppose we have a set of numbers, and we will call this set A, such that

$$A = \{1, 2, 3, 4, 5, 6\} \tag{1}$$

A set is "a collection of distinct objects", and in our case, mathematically, our sets will be made up from numbers. The natural numbers are considered to be the set of all integers, or whole numbers, that does not include zero, that are greater than or equal to 1; denoted by $\mathbb{N}$ and this set can be written as

$$\mathbb{N} = \{1, 2, 3, \dots\}. \tag{2}$$

Observe that all of the numbers in the set A are included in the set $\mathbb{N}$, and this is what it means for A to be a subset of $\mathbb{N}$. The natural numbers are called so because they are the numbers that are found in nature. Many mathematicians would explain this by saying you can count the natural numbers with your fingers, and this then excludes negative numbers, fractions and decimals, complex numbers and zero. Relating this to bells, bells are counted with the natural numbers. And since we don't have infinitely many bells in one tower, then we have a subset of the natural numbers.

We can relate the numbers to the bells that are being rung. In bell towers, the bells are usually arranged in a circle, from lightest to heaviest, clockwise. In the ringing world, the bell with the highest note is called the treble, and the bell with the lowest note is called the tenor. The number of bells in a tower can vary. Across the United Kingdom, many towers that campanologists regularly ring the bells at usually have 5 to 12 bells[1]. However, there are few that have 15 or even 16[1]; and there are many towers across the country that have as few as 3 bells (this excludes towers that have carillons in them, since these bells are rung by only one person, playing a contraption that resembles a piano, and following a music sheet as you would playing any other instrument). We label the bells according to how they are ordered in the circle and on the note they are tuned to; so the bell with the highest note is number 1, the next highest note is number 2, the next 3, and so on. The more bells there are in a tower, the bigger the circle has to be to accommodate the extra bells.

Apart from some towers having only 3 or 5 bells, it is very rare that odd numbers of bells are rung on their own; and this is for musical reasons. Every bell has a different note, and depending on how many bells there are relates to how many different notes there are. 15 bells cover two octaves. You are possibly thinking that this is not correct, since an **<u>oct</u>**ave relates to the number 8, and 8 multiplied by 2 is equal to 16. However, on 15 bells, the "front 8" bells cover one octave, and the "back 8" bells cover another octave. This means that the middle bell - number 8 - is included in two different octaves. This may be explained a little better by Figure 1 below:

Figure 1

Bell Number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Bell Note	D	C#	B	A	G	F#	E	D	C#	B	A	G	F#	E	D
	Octave								Octave						
	Holy and Undivided Trinity, Ossett, W. Yorks., England														

[2]

If a tower has 13 or 15 bells, this is usually because one of those bells is a flat note or a sharp note. For example, if a tower has 13 bells, then only 12 of those bells can be rung at the same time; the "spare bell" will be a second 6th bell. This is to accommodate how many campanologists are present. If there are only 8 campanologists present in the tower then 8 bells can still be rung and the extra bell makes it possible for the ringing to sound "complete" i.e. the notes of the different bells sound better. However, there's one example of a church having a 'straight ring' of 15 bells (this means that you can ring all 15 bells with the notes being in the correct musical order without any abnormalities), it's the only one that currently has this in the whole world, and that church is located in West Yorkshire, England [2].

4 Permutations and Cycles

We can now discuss the pure mathematical side of campanology. Earlier, we discussed that bells can be played using a contraption called a carillon, but when the bells are rung by teams of people they are hung on a wheel and attached to a rope; which is then pulled by the campanologist to make the bell ring. It was also mentioned that when ringing a carillon, the person playing the music follows music sheets just like any other instrumentalist would. However, when it comes to ringing a bell on a rope, campanologists follow patterns called "methods".

Suppose we have 6 bells; then the bells will always start ringing in the sequence

$$1, 2, 3, 4, 5, 6 \tag{3}$$

and if we have 12 bells they will always start in the sequence

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 \tag{4}$$

Remember that the bell with the highest note is labelled 1.

"... a permutation is the substitution of one arrangement for another, a rearrangement." [4] This means that by using permutations we can rearrange the bells into a different order, and every time we rearrange the bells and put all these rearrangements together to make a method. With 6 bells we can rearrange them 720 different ways. We can start with

$$123456 \tag{5}$$

And then swap the bells 1 and 2, 3 and 4, and 5 and 6; like so

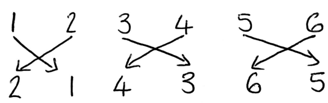
$$214365 \tag{6}$$

This permutation is represented by the following

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix} \tag{7}$$

This notation means that the number 1 goes to the "number 2 position" and the number 2 goes to the "number 1 position", number 3 goes to the "number 4 position" and so on. We can show this in a diagram, as shown in Figure 2:

Figure 2



In mathematics, we can swap any number with any other number. So we could have

$$123456 \tag{8}$$

go to

$$135246 \tag{9}$$

However, in campanology there's a rule - you cannot swap a bell with another that is not adjacent to it - which simply means that the bells have to be next to each other in the sequence for them to be able to swap position. This means that the above permutation in bell ringing is not possible unless some other permutations happen first. When we put many permutations together we create a method, and depending on the order of these permutations we can get many different methods. To find out how many different combinations of bells we can have can be done by some simple calculations. So for example, suppose we have 6 bells and we want to find out how many different combinations we can have - how many different ways can we order 6 bells? Well, first of all, we have 6 choices of which bell comes first; so suppose we put bell number 1 in the first position. For the next position we have a choice of the five remaining bells, for the next position we have a choice of the four remaining bells and so on; and we multiply these options together. So we conclude that the maximum number of different combinations of 6 bells we can have is 720; since

$$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6! = 720 \tag{10}$$

"The number of ways of arranging n unlike things all in a row is n!"[3]

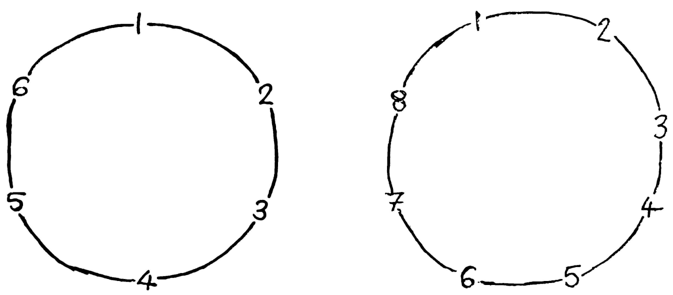
If n is any number and n has an exclamation mark next to it, then we call this "n-factorial" and we multiply all the numbers greater than zero up to and including the given n together.

As we discussed earlier, bells are arranged in a circle, clockwise, from the smallest to the largest (or the highest note to the lowest note). This is related to "cycles" in algebra.

*"If the numbers abcd are arranged clockwise round a circle in that order, then bcda, cdab, and dabc represent the same order of the letters, though they are different **linear** arrangements."*[4]

This relates well to the ringing circle, and this can be shown by the following diagrams, in Figure 3, for 6 and 8 bells

Figure 3



We can obtain the different permutations from the above quote by rotating the circles. Campanologists usually only move one bell like this by practising "called changes". Calling changes is where, usually, one member of the ringing team changes the order of the bells by swapping only two bells at one time. Suppose we want to swap bells 1 and 2, then we would "call" (i.e. shout to the rest of the team) *"treble to two"* (remembering that bell number 1 can also be called the 'treble') or, *"one and two"* or even "two lead" - these are all valid instructions to get bell number 1 following bell number two to get the order of bells from

$$123456 \tag{11}$$

to be

$$213456 \tag{12}$$

Now bell number 1 is following bell number 2 in the sequence, and the rest of the bells are in the same places as before - since we didn't call for them to change position. By carrying on making number 1 come after all the other bells, we get the pattern:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 5 & 6 \\ 2 & 3 & 4 & 1 & 5 & 6 \\ 2 & 3 & 4 & 5 & 1 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{matrix} \tag{13}$$

And then when we bring bell number 1 back to the front of the pattern we get the following:

$$\begin{array}{cccccc} 2 & 3 & 4 & 5 & 6 & 1 \\ 2 & 3 & 4 & 5 & 1 & 6 \\ 2 & 3 & 4 & 1 & 5 & 6 \\ 2 & 3 & 1 & 4 & 5 & 6 \\ 2 & 1 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array} \tag{14}$$

If we put these patterns together we retrieve the treble (bell number 1) "plain hunting" - meaning it moves to the back of the sequence and then to the front of the sequence again. When we have 6 bells all doing this at the same time then we get the following pattern:

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \\ 2 & 4 & 1 & 6 & 3 & 5 \\ 4 & 2 & 6 & 1 & 5 & 3 \\ 4 & 6 & 2 & 5 & 1 & 3 \\ 6 & 4 & 5 & 2 & 3 & 1 \\ 6 & 5 & 4 & 3 & 2 & 1 \\ 5 & 6 & 3 & 4 & 1 & 2 \\ 5 & 3 & 6 & 1 & 4 & 2 \\ 3 & 5 & 1 & 6 & 2 & 4 \\ 3 & 1 & 5 & 2 & 6 & 4 \\ 1 & 3 & 2 & 5 & 4 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array} \tag{15}$$

We can examine this pattern further and say that each bell is moving the same as every other bell, only each bell starts in a different position to the others. We have already mentioned "methods", and this is how they are performed - with the bells making the same pattern of permutations, but starting in different positions, knitting together to make a musical tune. The different methods (or tunes) are usually called after different place names, just because this is what the first people who named the first tunes decided to start doing and no one has ever deviated from this decision.

Going back to factorials and how many different changes we can ring on the bells, campanologists sometimes ring for extended periods of time. You may have heard of a 'peal of bells'. In the ringing world, peals are rung for special occasions or just for fun. For a peal to be called 'true', the team of campanologists needs to ring at least 5000 different changes. You may be questioning what happens if only 6 bells are being rung - since the 5000 changes have to be different. Well, you can only ring 720 different (unique) changes on 6 bells, so when ringing a peal of at least 5000 changes you can repeat any changes, but only after ringing those 720 unique changes first. These changes can be achieved by ringing one pattern (method) or by combining two or more methods. This can also be said about ringing a peal on 5 bells. However, we don't come across this problem on 7 or more bells, since the maximum number of changes that can be rung on 7 bells is 5040 (and on even more bells, this number just gets larger).

5 Conclusions

In this paper we have explored how mathematics, in particular algebra, applies to campanology and ringing the changes. Permutations and cycles and some of the more pure side of mathematics can be applied to something many people recognise as a sound of joy and celebration, or sadness and sorrow. Many people who ring the church bells don't necessarily know how their hobby relates so well to mathematics, but next time you are near a church with bells, take a moment to remember what we have discussed in this paper, and how complicated it may get with all the different combinations of numbers and bells.

References

[1] Ronald H Dove. *Dove's Guide*. 21st February 2019. URL: `https://dove.cccbr.org.uk/home.php`.

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[3] Clement V. Durell. *Advanced Algebra*. 1932, pp. 1, 2, 3, 4, 5.

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[5] Burkard Polster and Marty Ross. *Ringing the Changes*. 21st February 2019. URL: `https://plus.maths.org/content/ringing-changes`.

[6] Bach Van Der Sluijs. “Change Ringing”. In: *Mathematics Institute, Leiden University* (June 2016).