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A4 7 Brachistochrone Problem in a Central Force Field

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Abstract

The brachistochrone problem, which seeks the curve of fastest descent between two points under gravity, is a foundational problem in the calculus of variations. In this paper, we revisit this problem and extend the analysis to a non-uniform gravitational field governed by an inverse-square central force field. A corresponding *time functional* is constructed for this system and numerically evaluated for four candidate curves. The result identifies the optimal path among these candidates and provides a quantitative assessment of how the optimal trajectory deviates from the cycloid when the accelerating force varies with distance.

Introduction

The brachistochrone problem, posed by Johann Bernoulli in 1696, asks for the path $y(x)$ between points A and B that minimises the time functional $T[y]$ for a particle under gravity [1]. The classical solution is a cycloid, derived by applying the Euler–Lagrange equation to the time functional defined in a uniform gravitational field [2].

However, the assumption of a uniform field is an approximation. In reality, the gravitational force is non-uniform and follows an inverse-square law [3]. We investigate the brachistochrone problem within a true $1/r^2$ central-force regime, exemplified by the path of quickest descent for a probe falling towards a planetary body. The total descent time is compared across four candidate curves, demonstrating that the cycloid ceases to be the optimal solution in this scenario.

Theoretical Framework

We consider a particle released from rest at the point $A(x_A, y_A)$ and falling toward a central mass M under an inverse-square gravitational field. The gravitational potential U is

$$U(r) = -\frac{GMm}{r}, \tag{1}$$

where $r = \sqrt{x^2 + y^2}$. Since the particle starts from rest, the total energy E_{total} is conserved and equal to the initial potential energy,

$$E_{\text{total}} = U(r_A) = -\frac{GMm}{r_A}. \tag{2}$$

The kinetic energy at a point along the path is obtained from $K = E_{\text{total}} - U(r)$. Hence, the velocity as a function of position satisfies,

$$v(r) = \sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_A}\right)}. \tag{3}$$

This expression is valid for $r < r_A$, consistent with motion toward the attracting body.

The differential arc length $\mathrm{d}s$ along the curve $y(x)$ is given by $\mathrm{d}s = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \sqrt{1 + (y')^2}\,\mathrm{d}x$, and the time of descent along the path from A to B is $T = \int \frac{ds}{v}$. Substituting this yields the functional

$$T[y] = \int_{x_A}^{x_B} \frac{\sqrt{1+(y')^2}}{\sqrt{2GM\left(\frac{1}{r}-\frac{1}{r_A}\right)}}\,dx. \tag{4}$$

This reduces to the classical brachistochrone integral under the uniform-field approximation $\frac{1}{r} \approx \frac{1}{r_A} - \frac{y}{r_A^2}$.

Numerical Analysis and Results

The improper integral in Equation (4) is evaluated numerically using R's `integrate()` function [4]. To simplify computation, we set $GM = 1$, a standard non-dimensionalisation choice that defines natural units for the problem. This scaling removes extraneous constants while preserving the relative descent times between paths. The resulting times are expressed in the natural time units and reported as "seconds".

The defined start and end points are $A = (0, -4)$ and $B = (\pi, -6)$. This coordinate choice models a path that is far from the central mass at the origin, preventing the potential from diverging, which could occur if paths crossed at $r = 0$. The x-coordinate interval chosen is convenient for comparison with the classical cycloid, which is naturally parametrised by π.

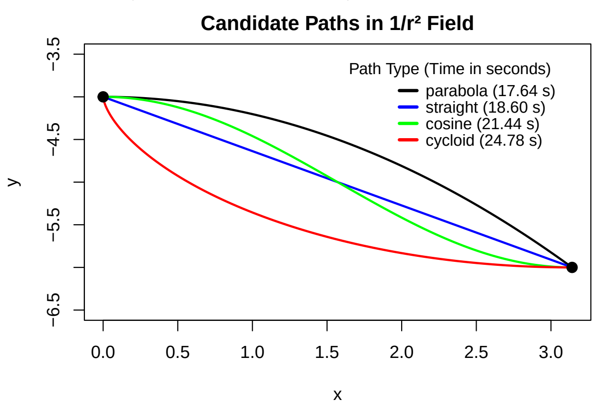

Figure 1: Comparison of descent times in the non-uniform $1/r^2$ central-force field. Coordinate axes in metres; path types ordered from fastest to slowest.

Figure 1 illustrates the four candidate paths that are evaluated: a straight line, representing the shortest distance; a parabola, providing a simple non-linear alternative; a cosine wave, exhibiting a non-zero initial slope; and the classical cycloid, which is the solution to the uniform-field problem. The computed descent times are compared quantitatively.

Discussion

The paths in Figure 1 reveal a result that fundamentally contradicts the classical uniform field solution. In the classical problem, the cycloid is always the path of fastest descent. However, in the present analysis, the cycloid produces the slowest descent time of 24.78 s, while the parabola yields the fastest with a time of 17.64 s.

This reversal arises from the non-linear nature of the inverse-square central force field. The classical cycloid is designed for rapid acceleration, in a uniform field. In contrast, under a $1/r^2$ potential, the particle begins far from the origin, where the potential is weaker. The parabolic trajectory proves more efficient by maintaining a shallower slope, which better balances the overall path distance with the spatially varying acceleration compared to the cycloid's steep initial drop.

The successful convergence of this numerically evaluated functional demonstrates the effectiveness of this approach for solving non-trivial variational problems in physics.

Conclusion

This paper presented an analysis of the brachistochrone problem by replacing the standard uniform field with one that is non-uniform. The key finding is that the optimal path in a non-uniform field deviates from the classical cycloid, with the parabolic trajectory producing the fastest descent. This result highlights that the brachistochrone solution is critically dependent on the form of the potential field and confirms the necessity of numerical methods for addressing non-uniform gravitational systems.

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