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P3_7 Charged particles in a magnetic field - an integrator

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Abstract

We investigate the accuracy of using a simple numerical integrator (Euler Method) to find how a charged particle moves in a magnetic field. We find that the error depends on the choice of time step, but is not negligible for even small values, as such it is probably not the best method for this.

Introduction

Numerical integrators can be useful ways to approximate real life situations. In this paper we investigate a simple form; The Euler Method [1]. We evaluate the accuracy of this by applying it to a specific situation - charged particles gyrating in a magnetic field. In reality, the kinetic energy of the particle is constant, but due to the error of the integrator it increases over time. We will use this increase to quantify the error.

Theory

We first need to find equations to describe the motion of charged particles in a magnetic field. Since this paper is more about numerical integration than deriving the expressions, we will just quote them Eq.(1). We define the magnetic field along the z-axis only, and that our particle has some velocity with a component in this direction and also perpendicular. The components of acceleration in each direction a are [2]:

$$a_x = \frac{qB}{m}v_y \qquad a_y = -\frac{qB}{m}v_x \qquad a_z = 0 \tag{1}$$

We also know that the kinetic energy of the particle at any given time is:

$$E_k = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2) \tag{2}$$

We also choose units such that the charge $q = 1$, the magnetic field $B = 1$ and mass $m = 1$. v is the velocity and $v_\perp$ is the velocity perpendicular to the magnetic field.

Method

As briefly mentioned we will be using the Euler Method. This means we find the acceleration from Eq.(1), then multiply by a finite time step dt to find the velocity and position [1]:

$$a_x = \frac{qB}{m}v_y \qquad a_y = -\frac{qB}{m}v_x$$

$$v_x = a_x dt \qquad v_y = a_y dt$$

$$x = v_x dt \qquad y = v_y dt$$

We then repeat this process, putting the new value back into the acceleration expressions i.e. perform an iteration. We note that since $a_z = 0$, v_z will stay constant and our trajectory will move along the z-axis at a constant rate. The choice of dt is related to how accurate our result will be, since we are approximating a curve as a repeated series of small straight lines (Figure 1). It could be chosen to arbitrarily accurate precision, but this makes the process take longer. However since we are running a simple case with not too small dt we don't need to consider this.

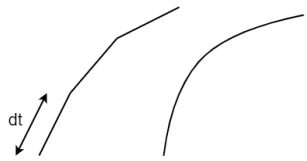

Figure 1: Diagram illustrating why the choice of time step is important. The left is our approximation and the right is the true curve. If we had a smaller time step, so more lines, it would fit the curve better.

Results & Discussion

We code this process into a loop, such that it will repeat as many times as we want. We record the position each iteration so that we can plot this at the end. We define the initial velocities $v_z = 1, v_\perp = 1$. For this limited situation we choose only a few different values of dt to illustrate how the inaccuracy changes, rather than trying to quantify how it changes in general.

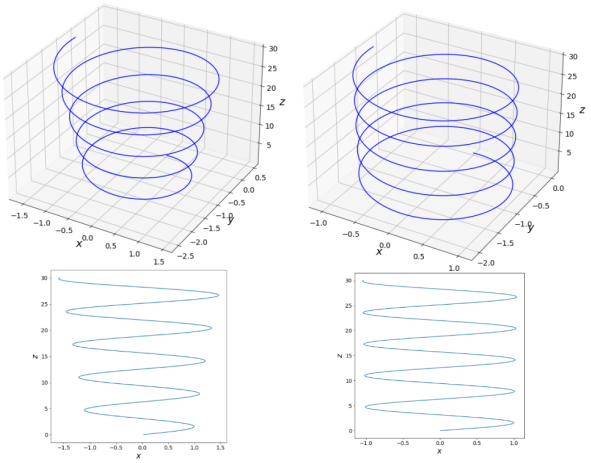

Figure 2: Example particle trajectories in a magnetic field as computed by the integrator. Left: $dt = 0.03$ with 1000 iterations. Right: $dt = 0.003$ with 10,000 iterations. A 3D (top) and 2D (bottom) plot are shown for each.

We see that for a smaller dt, the particle doesn't "spiral out" as much. This is good because the spiralling out means that the particle is gaining energy which is the inaccuracy introduced by this method. We can calculate the kinetic energy of the particle Eq.(2), at each iteration, which is slowly increasing due to error being introduced, and plot a graph of how the energy (which relates to error) increases over time. Due to our choice of units the initial kinetic energy is just $E_k = 1$, and anything above this is the error on top. This means the gradient of the graph is just the error introduced per step.

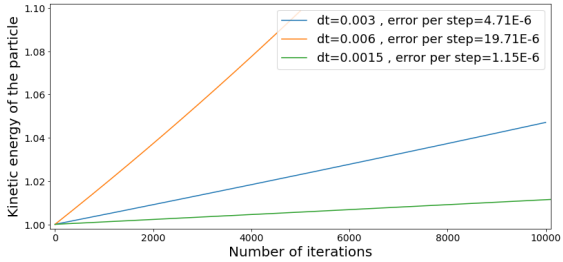

Figure 3: Kinetic energy of the gyrating particle over time (step on the integration)

Technically the error exponentially increases, but since each error is of order $\sim 10^{-6}$ and it doesn't run for too long, they don't really compound so we can approximate as a linear increase. This lets us find the approximate error per step which is shown in the legend. We also note that the even for an arbitrarily small dt, the error will never be zero. In general, when we apply this method we must weigh up how much error we are willing to accept, versus how much time we want the simulation to run.

Conclusion

We find that, as expected, the error is dependent on the time step. If we choose a smaller time step, our error is less, however it is still not a negligible error even for a very small step, and for many iterations this would build up. In the future we may wish to compare the errors introduced by this method, and the errors from other methods (such as leapfrog) to see which would be a better choice for a given situation.

References

[1] P.A. Tipler and G. Mosca. *Physics for scientists and engineers : with modern physics.* W.H. Freeman, New York, 6th edition, 2008.

[2] Steve Milan. Space Plasma Physics (full notes). University of Leicester, PA4603, 2019. Course taken 2022.