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P2_1 A PREMium Model

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Abstract

This paper verifies that the radial density equations outlined in the Preliminary Reference Earth Model (PREM) are accurate by using them to calculate a value of 5.973×10^{24} kg for the mass of the Earth. This was found to be only 0.01% different from the known value of the Earth's mass.

Introduction

The Preliminary Reference Earth Model (PREM) was developed in 1981 and is still a leading model in seismology worldwide today [1]. The one-dimensional model describes the Earth's properties, such as density, as a function of radial distance. The density functions from the PREM paper can be found in Table. 1 below.

Radius (km)	Densities (kg/m^3)
0.0000 - 1221.5	$\rho_1 = 13088.5 - 8838.1x^2$
1221.5 - 3480.0	$\rho_2 = 12581.5 - 1263.8x$ $-3642.6x^2 - 5528.1x^3$
3480.0 - 5701.0	$\rho_3 = 7956.5 - 6476.1x$ $+5528.3x^2 + 3080.7x^3$
5701.0 - 5771.0	$\rho_4 = 5319.7 - 1483.6x$
5771.0 - 5971.0	$\rho_5 = 11249.4 - 8029.8x$
5971.0 - 6151.0	$\rho_6 = 7108.9 - 3804.5x$
6151.0 - 6346.6	$\rho_7 = 2691.0 + 692.4x$
6346.6 - 6356.0	$\rho_8 = 2900$
6356.0 - 6368.0	$\rho_9 = 2600$
6368.0 - 6371.0	$\rho_{10} = 1020$

Table 1: A table of the density functions at their corresponding radii where R_E is the Earth's radius, $x = \frac{R}{R_E}$, $R_0 = 0$ km, $R_1 = 1221.5$ km, $R_2 = 3480.0$ km, etc. [1]

Further, a graph of the density functions against radial distance was plotted in Fig. 1.

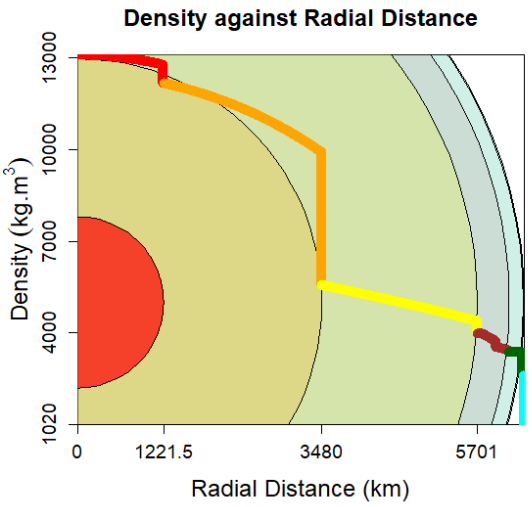

Figure 1: The line represents the density as a function of radial distance and the background circles represent the Earth's radial regions (core, mantle, crust, etc.)

Method

We investigated the constraints of the model by evaluating that the density equations integrated across the entire Earth's volume sum to the known value of the Earth's mass. The total mass of the Earth can be found by integrat-

ing the density equations from the PREM model with respect to radius from their lower limit to their upper limit. This will yield the mass values between each radial region of the Earth and summing all these masses together will yield the total mass of the Earth.

The mass, M_i for each of the density equation radial regions can be found by integrating the densities from R_{i-1} to R_i:

$$M_i = \int_{R_{i-1}}^{R_i} \rho_i(x) dV \tag{1}$$

where $dV = 4\pi R^2 dR$ due to radial symmetry [3].

The density equations take the general form of $\rho_i = A_i + B_i x + C_i x^2 + D_i x^3$, where $x = R/R_E$. Substituting these values into Eq. (1) gives:

$$M_i = 4\pi \int_{R_{i-1}}^{R_i} \left[A_i + B_i \left(\frac{R}{R_E} \right) + C_i \left(\frac{R}{R_E} \right)^2 + D_i \left(\frac{R}{R_E} \right)^3 \right] R^2 dR \tag{2}$$

This solves to be:

$$M_i = \left[\frac{2\pi}{3R_E^3} D_i (R_i^6 - R_{i-1}^6) + \frac{4\pi}{5R_E^2} C_i (R_i^5 - R_{i-1}^5) + \frac{5\pi}{5R_E} B_i (R_i^4 - R_{i-1}^4) + \frac{4\pi}{3} A_i (R_i^3 - R_{i-1}^3) \right] \tag{3}$$

As there are 10 different density equations, the total mass of the Earth can be found to be:

$$M_E = \sum_{i=1}^{10} M_i \tag{4}$$

Results

The values from Table 1 for $i = 1$:
$A_1 = 13088.5$ kg/m^3, $B_1 = D_1 = 0$ kg/m^3, $C_1 = 8838.1$ kg/m^3, $R_0 = 0$ km, and $R_1 = 1221.5$ km. Substituting these values into Eq.(3) gives:

$$M_1 = \left[\frac{4\pi}{5R_E^2} (8838.1)(R_1^5 - R_0^5) + \frac{4\pi}{3} (13088.5)(R_1^3 - R_0^3) \right] \tag{5}$$

Thus, $M_1 = 9.843 \times 10^{22}$kg.

The mass for each radial region was calculated following the same example as above.

$M_1 = 9.843 \times 10^{22}$ kg	$M_2 = 1.841 \times 10^{24}$ kg
$M_3 = 2.940 \times 10^{24}$ kg	$M_4 = 1.153 \times 10^{23}$ kg
$M_5 = 3.334 \times 10^{23}$ kg	$M_6 = 2.899 \times 10^{23}$ kg
$M_7 = 3.235 \times 10^{23}$ kg	$M_8 = 1.382 \times 10^{22}$ kg
$M_9 = 1.587 \times 10^{22}$ kg	$M_{10} = 1.560 \times 10^{21}$ kg

Table 2: A table describing the mass for each radial region of the Earth

The predicted Earth mass from the PREM model can then be calculated using Eq. (4) which gives a value of $M_E = 5.973 \times 10^{24}$ kg.

Discussion

The known value of the Earth's mass is 5.9724×10^{24} kg [2]. The value calculated using the PREM model is therefore only 0.01% larger than the known value. The result derived from the PREM model is therefore an accurate measure of the Earth's mass. The error may be attributed to the Earth being an oblate spheroid rather than perfectly spherical, which is assumed in this paper to simplify the calculations.
The derived mass value implies that the model's density equations are accurate, which is significant as the PREM model is used for a variety of computations within seismology, such as for synthetic seismographs.

References

[1] `https://lweb.cfa.harvard.edu/~lzeng/papers/PREM.pdf` Dziewonski, Adam M. and Anderson, Don L. (1981) Preliminary reference Earth model. [Accessed 12 October 2021]

[2] `https://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html` [Accessed 12 October 2021]

[3] P. A. Tipler, Physics For Scientists and Engineers (W.H Freeman and Company, New York, 2008), 6th Edition, Chapter 9, pages 289 - 355.