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A5_5 Comparing Energy of Free-fall in a Schwarzschild Space-time to Special Relativity

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Abstract

We compared energy in a warped Schwarzschild space-time (SST) to special relativistic energy in flat Minkowski space-time (MST) by equating the Lorentz transformed energy of a particle in MST to the energy of a free-falling particle in SST. We develop a relation where a coefficient translates free-fall velocities in SST to velocities in MST. Higher velocities needed in SST demonstrate the existence of a gravitational potential well, one that does not match the values of Newtonian Potential energy. Moreover, a particle in SST of a Plank mass at 2 Schwarzschild radii free-falls at $\frac{\sqrt{2}}{2}$ c to have an equivalent energy of a stationary particle in MST.

Introduction and Theory

We will be comparing energy an object has in warped space-time to its energy in special relativity. Special relativity only applies to a 'special' case where space-time is flat Minkowski space-time (MST). Special relativistic effects occur when an object is in a moving time-frame with a relative velocity, v_η, to the observers time-frame. We calculate the effects by applying a Lorentz transformation to the moving object's time-frame. We can represent this as

$$X' = LX. \tag{1}$$

Where the object's time-frame, X, is

$$X = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}, \tag{2}$$

L for a velocity in the x direction is

$$L = \begin{pmatrix} \gamma & -\gamma\beta_\eta & 0 & 0 \\ -\gamma\beta_\eta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3}$$

the apostrophe denotes the observers measurement of X, the Lorentz factor, γ, is

$$\gamma = \frac{1}{\sqrt{1-\beta_\eta^2}} \tag{4}$$

and β_η is the v_η as a fraction of c (speed of light)[1].

Observing an object at non-identical distances from a gravitating mass can also cause gravitational relativistic effects. We can think of it as special relativistic effects of an accelerating time-frame where the acceleration is due to the bending of space-time done by the mass. An example would be Schwarzschild space-time (SST). The bending of SST is caused by a non-spinning black hole with no charge.

In order for us to compare energy in MST to SST we first need the equation of motion of a free-falling test particle, P_s, with negligible mass in SST,

$$\frac{E^2}{1-\frac{r_s}{r}} = \frac{1}{1-\beta_s^2}. \tag{5}$$

Where E is the energy of the particle, r is the particle's radial distance from the singularity, β_s is the velocity of the particle as a fraction of c and r_s is the Schwarzschild radius [2]. We can now make the comparison by substituting E with the Lorentz transformed energy, E', of a test particle, P_η, traveling in MST, where

$$E' = \gamma m_0 c^2 \tag{6}$$

and m_0 is P_η's rest mass. After substituting equation (6) into equation (5), with E' as E, we rearrange to get the relation

$$1-\beta_s^2 = A\left(r\right)\left(1-\beta_\eta^2\right) \tag{7}$$

$$A\left(r\right) = \frac{1-\frac{r_s}{r}}{m_o^2 c^4}, \tag{8}$$

illustrating the β_s needed for P_s to have equivalent energy to P_η.

Discussion and Conclusion

From plotting equation (7), Figure 1, we can see the shape of the relation. To match the energy of a stationary P_η (L becomes the identity matrix) P_s has to travel at

$$\beta_s = \sqrt{1-A\left(r\right)}. \tag{9}$$

For the parameters we set in Figure 1 equation (9) has the value $\frac{\sqrt{2}}{2}$ c.

Naturally, the relation with any physically possible value for the parameters will go through the point (1,1) on the plot as it is impossible to travel faster than c.

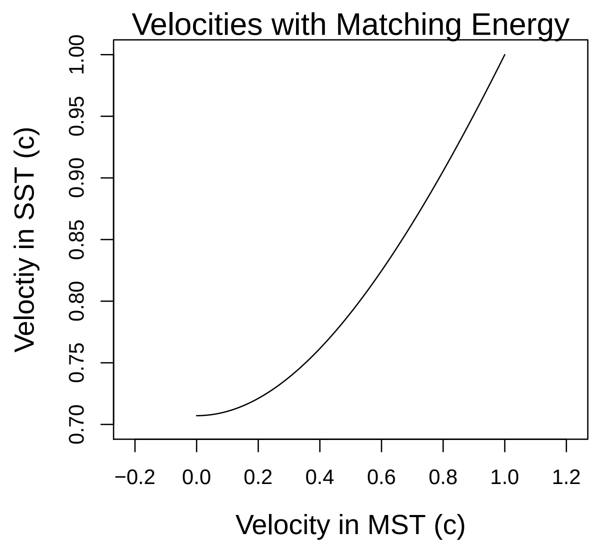

Figure 1: Displays the relationship between β_s and β_η where r=2r_s and m_0 is a Plank mass.

Figure 1 indicates that at all velocities except c β_s, is a higher value than β_η. These higher velocities needed in SST to match energy in MST demonstrates the gravitational potential well caused by warping in SST. However, once analysed, we realised kinetic energy from the velocity making up for the potential well does not match Newtonian values for gravitational potential energy, thus illustrating the difference between relativistic and non-relativistic theories.

References

[1] Raine, D. and Thomas, E. (2006). Black Holes. London: Imperial College Press.

[2] Steane, A. (2012). Relativity made relatively easy. Oxford: Oxford University Press.