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P2_7 Power Curves and Gear Ratios in Bicycles

J. Anand, A. Buccheri, M. Gorley, I. Weaver

Department of Physics and Astronomy, University of Leicester, Leicester, LE1 7RH.

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Abstract
This article investigates how the power exerted by a cyclist varies with cyclist speed, and gear ratio. From Hill's relation, both relations are determined and plotted. Suggestions are made on how these results might be useful to recreational and competitive cyclists.

P2_2 Sports Science

Introduction

Gears in bicycles are often misused. Many people have experienced that pedalling in a high gear from stationary is very difficult, and produces very little acceleration. Equally, a low gear at high speeds is inefficient. This article explores the relationship between bicycle gearing, and the cyclist power output.

Power Functions

The force exerted by any engine is always applied in 2 different areas. The first is accelerating engine components, and the other is in the system on which you want to do work. For a cyclist, the first area is the person's legs and the pedal crank, and the second is the torque exerted on the bicycle pedals. In the limit of very slow leg movement (high resistance), the force needed to accelerate the legs tends towards zero, and maximum force, F_0 is applied to the pedals. As the rate of movement increases, the energy taken to move the legs at high speeds increases until the cyclist is pedalling at the maximum rate, ω_0 where all of the force is going into accelerating the legs, and zero force is applied to the pedals. The relationship between contraction speed v and force F exerted by muscles is given by Hill's relation [1].

$$(F+a)(v+b)=c$$

where a, b and c are constants with units of force, velocity and power respectively. For this article, it is more useful to think of pedalling rates than contraction velocity (as in Hill's relation), so we substitute $v=\omega r$ where r is the pedal crank arm length. Originally, Hill used parameters for the mechanics of muscle fibres to determine the unknowns in this equation. However, since the cyclic pedalling motion is not as well studied as basic linear contraction, values for a, b and c must be found from the special cases. That it, at $\omega=0$ where $F=F_0$, and at $F=0$ where $\omega=\omega_0$.

$$b(F_0+a)=c \quad (1) \qquad a(\omega_0 r+b)=c \quad (2)$$

These 2 equations alone are not enough to determine values for the 3 unknowns. However, by considering the maximum power output for a cyclist, we have one more equation. We assume the maximum power output occurs where when the ratio $\omega/\omega_0=n$, where n is a dimensionless ratio. We can use Hill's equation to find the power and then differentiate it.

$$P=F\omega r=\frac{c\omega r}{\omega r+b}-a\omega r \qquad (3)$$

$$\frac{dP}{dr}=\frac{bc}{(\omega r+b)^2}-a$$

which is zero for $\omega=n\omega_0$, the point of maximum power output. Rearranging for c gives

$$c=\frac{a(b+n\omega_0 r)^2}{b} \qquad (4)$$

Equating (1) and (2), and rearranging for b gives

$$b(F_0+a)=a(\omega_0 r+b)$$

$$b = \frac{a\omega_0 r}{F_0} \quad (5)$$

Equating (1) and (4), and substituting in (5) gives

$$a(b + n\omega_0 r)^2 = b^2(F_0 + a)$$

$$a = \frac{F_0 n^2}{1-2n} \quad (6)$$

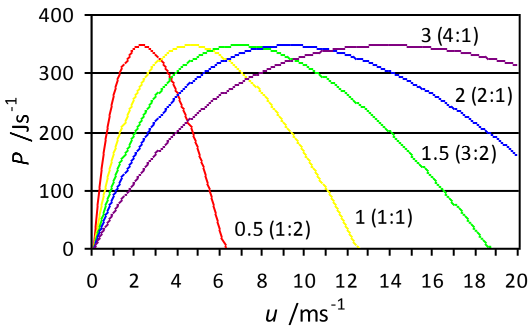
By estimating values for F_0, ω_0 and n, a, b and c can be found using equations (6), (5) and (1) respectively.

Gearing

Bicycle gears are used to help a cyclist maintain a high power-output across a range of cycling speeds. Transmission to the pedal drive shaft to the rear wheel is done via a chain, so the gear ratio R is given as the ratio of spikes on the pedal cog to the spikes on the wheel cog. A ratio of 0.5 would mean two turns of the pedal shaft would turn the bicycle wheel once. The pedal angular velocity ω is related to the wheel angular velocity ω' by multiplying by R, and this is related to the bicycle speed u by further multiplying by D, the tyre diameter.

$$u = D\omega' = DR\omega \quad (7)$$

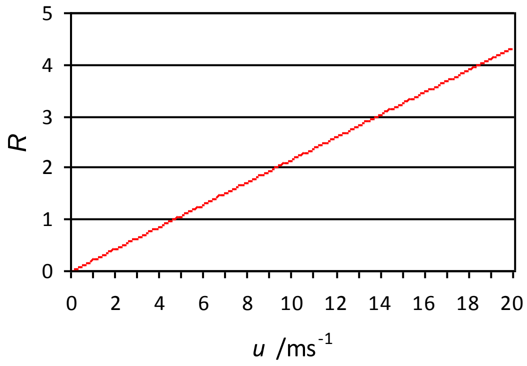
Substituting (7) into (3) for ω, and plotting over a range of u for different R gives a plot of the power curve for different gear ratios. Parameters are estimated as follows. A fit, recreational cyclist typically ought to be able to limited range leg-press 400lbs [2]. In terms of force, this equates a F_0 of approximately 800N. ω_0 is approximately 3 revolutions per second. n is the ratio of ω to ω_0 for maximum power output, which is roughly 0.37 for non-athletes [3]. r is 170mm and D is typically 26in [4, 5].



Graph 1 shows the power curves for different gear ratios.

The peaks of these power curves give the optimal speed to accelerate in that gear ratio. The optimal gear ratio as a function of speed is simply given by

$$R = \frac{u}{Dn\omega_0} \quad (8)$$



Graph 2 shows the functional form of (8).

Discussion and Conclusion

Graphs 1 and 2 show how power curves shift with different gear ratios, and demonstrate why it is important to maintain the correct gear ratio to accelerate. Graph 1 shows this is most important for slow speeds. For example, at 5ms^{-1}, the power output in the correct ratio (1:1) is almost double that in the lower (1:2) ratio.

Bicycles only have a set number of gears, but by programming a speedometer with the ratios for a given design, and any other required parameters, it could use graph 2 to advise the rider on the optimal gear available for his current speed. This would be of particular use to inexperienced cyclists who might not yet have a "feel" for the correct gear.

References

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[2] Leg Press test, *http://www.brianmac.co.uk/lpress.htm*

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[4] Bicycle Crank Length, *http://bicyclecranklength.blogspot.com/*

[5] Tyre Sizing, *http://www.sheldonbrown.com/tire_sizing.html*