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P2_3 A Tapered Tower

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Abstract

This paper evaluates the possibility of building a space elevator from a mechanical standpoint; calculating the yield stress and required density of a homogeneous material that such a structure would ideally need. It is concluded that a tapered tower made from carbon nanotubes may potentially be feasible.

P2_1 General Physics

Introduction

A *"space elevator"* can be thought of as a taut, rigid structure that stretches from the Earth's surface up to a point where its centre of gravity is at a height of 35,000 km [1], the orbital height of a geostationary satellite. Attached at the top of the structure is a counterweight to keep the structure taut. Due to the relative ease of ascending such a tower compared to conventional launch methods, the concept has been viewed as a potential launch mechanism.

By staying upright the elevator does not orbit at a uniform velocity due to the change in centripetal acceleration along its length. As a result, the tension in the elevator varies along its length, reaching a maximum at its centre of gravity. This change in tension, along with the sheer height of the finished structure poses serious problems in designing and building a stable structure.

To begin with, it will be assumed that the elevator will be built out of a single, homogeneous cable.

A Free-Standing Cylindrical Cable

Consider the case of a cylindrical cable of constant cross-sectional area in orbit. As shown in Figure 1, the main forces that act on an element of the cable in equilibrium are the tension in the cable, the centripetal force pushing it away from the Earth and the cable's weight pulling it towards the Earth. Here, the tension can be defined as: *AdT*, where *A* is the cable cross sectional area and *T* is the force per unit area acting on the cable.

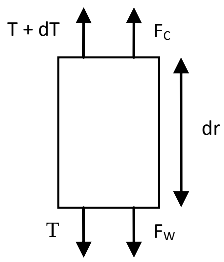

Fig 1: A cross section of an element of the elevator cable showing the forces acting on it. Here, F_W is the combined weight of the cable at that point whilst F_C is the centripetal force acting on the cable at that point.

Using standard formulae [2] describing gravitation and centripetal acceleration, it can be shown that equating the forces in equilibrium gives

$$AdT = \frac{GMA\rho dr}{r^2} - A\rho\omega^2 rdr \quad (1)$$

where ω and M are the rotational velocity and mass of the Earth, and r is the length of the cable from the centre of the Earth to its peak H. Here ρ is also the density of the cable. Rearranging and substituting the appropriate equivalent for ω gives a differential equation that describes the rate of change in tension as a function of radial distance from Earth

$$\frac{dT}{dr} = GM\rho\left[\frac{1}{r^2} - \frac{r}{R_g^{\ 3}}\right] \quad (2)$$

It can be demonstrated that the tension is at a maximum at R_g, (radius of geostationary orbit).

It is also known that the tension in the cable is zero at either end, so the boundary conditions for this equation are *T(R)=0* (where

R is the radius of the Earth) and $T(H)=0$. Integrating this equation between the limits $R \rightarrow R_g$ and $R_g \rightarrow H$ will therefore both give an expression that gives the tension at R_g. For instance, the tension at R_g integrated between R and R_g is given as:

$$T(R_g) = GM\rho\left[\frac{1}{R} - \frac{3}{2R_g} + \frac{R^2}{2R_g^{\ 3}}\right] \quad (3)$$

It can be shown that equating the two integrals $R \rightarrow R_g$ and $R_g \rightarrow H$, then rearranging for H gives the height of the elevator as approximately 144,000 km.

Substituting values for steel [3] into equation (3) gives a maximum tension per unit area of approximately 382 GPa, far in excess of the yield strength of steel. Similarly it can be shown that even advanced materials such as carbon nanotubes are inadequate.

Model: A Tapered Cable

Consider a different structure where the tension per unit area remains the same, but the area is allowed to vary as a function of *r*. That is, the tension in the cable can now be expressed as *TdA.* Following the same method as before, resolving the forces at equilibrium gives

$$\frac{dA}{A} = \frac{gR^2\rho}{T}\left[\frac{1}{r^2} - \frac{r}{R_g^{\ 3}}\right]dr \quad (4)$$

where g is the gravitational acceleration at the surface of earth. Note that the argument inside the brackets is the same as in equation 2. Integrating the above equation therefore gives the cross-sectional profile:

$$A(r) = A_s \exp\left[\frac{gR^2\rho}{T}\left\{\frac{1}{R} + \frac{R^2}{2R_g^{\ 3}} - \frac{1}{r} - \frac{r^2}{2R_g^{\ 3}}\right\}\right] \quad (5)$$

where A_s is the cross-sectional area at the surface of earth.

To find the height of such a structure the total weight of the structure needs to be equated to the total centripetal force pushing it outward. Due to the nature of the integrals this has to be done numerically.

To be in equilibrium, the tapered cable would have a length of approximately 144,000 km. That is, tapering the cable has no effect on the overall length of the cable. However, the trade-off between tension and area has another effect. This is demonstrated by dividing $A(R_g)$ by $A(R)$ to show the ratio between the area at the widest point and the area at the bottom. Substituting values for steel gives a taper ratio of 1.6×10^{33}. Even with this shape it would be dimensionally impractical to make a cable out of steel.

However, another possibility exists in using carbon nanotubes [4]. Substituting the relevant data into equation (5) this time gives a taper ratio of ~ 1.6. A tapered tower made out of carbon nanotubes would only need to be 1.6 times wider at its widest point than at the surface. This is much more structurally feasible.

Conclusions

Constructing a space elevator would be a mammoth undertaking, due to the length it would have to be. The magnitude of the tension in a cylindrical model makes it impractical. Tapering the cable would offer the advantage of having a uniform tension throughout the elevator, with the length being the same as the cylindrical one. However, steel would still be unsuitable due to the large taper ratio required.

That being said, an elevator made from a tapered carbon nanotube cable could potentially be viable, given the high tensile stress and low density of the material gives a modest taper ratio.

References

[1] BNSC: "Where do Satellites Orbit?", *http://www.bnsc.gov.uk*

[2] P. Tipler, G. Mosca, *Physics for Scientists and Engineers* (5^{th} Ed.), 2003

[3] SiMetric, *http://www.simetric.co.uk*

[4] S. Iijima, *Helical microtubules of graphitic carbon*, Nature, **354**, 56–58, 1991