Journal of Physics Special Topics

P1_3 The Double Pendulum and Lunar Seismometry

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November 12, 2014

Abstract

This paper aims to show that through the properties of the Laplace transformation into 's' space, the coupled differential equations of the double pendulum, as shown in figure 1 [1] can be solved with relative ease and show interesting properties and applications of the system for variable values of gravity. These include lunar seismometry and consideration of similar experiments on other astronomical bodies.

Definitions

The formal definition of the Laplace transformation [2] (Equation 1) involves an integration similar to that of the Fourier transform whereby a function can be transformed into 's' space, with the advantage that differential equations are reduced to simple algebra. The transform into 's' space is

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \hat{F}(s), \quad (1)$$

whilst the reverse transformation back into 't' space is managed through transform pairs [3].

The differential equations that describe the double pendulum in figure 1 are shown in equations 2 and 3:

$$m_2 {L_2}^2 \frac{d^2\theta_2}{dt^2} + m_2 L_1 L_2 \frac{d^2\theta_1}{dt^2} + m_2 L_2 g \theta_2 = 0 \quad (2)$$

$$(m_1 + m_2){L_1}^2 \frac{d^2\theta_1}{dt^2} + m_2 L_1 L_2 \frac{d^2\theta_2}{dt^2} + (m_1 + m_2) L_1 g \theta_1 = 0. \ (3)$$

Throughout this paper, the use of the linearity of the Laplace transform and its general behaviour with respect to differentials (shown in equation 4) will be used in order to condense notation:

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} \frac{df(t)}{dt}\, dt = [e^{-st} f(t)]_0^\infty + s\int_0^\infty e^{-st} f(t) dt = -f(0) + s\mathcal{L}\{f(t)\} = s\hat{F}(s) - f(0), \ (4)$$

Diagrams

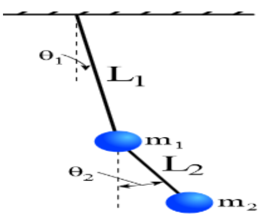


Figure 1 [1]: The double pendulum with labelled parameters.

In this paper, the following values and initial conditions will be used:

Values (Arbitrary units)	Initial conditions (Degrees)
$m_1 = 3$	$\theta_1(0) = 1$
$m_2 = 1$	$\dot{\theta}_1(0) = 0$
$L_1 = L_2 = 16$	$\theta_2(0) = 0$
g = variable	$\dot{\theta}_2(0) = -1$

Application

Equations 2 and 3 will not be derived; they will just be taken as general form [4]. Once the values and initial conditions detailed above are substituted in, equations 2 and 3 can be transformed through the use of equation 4. Subsequent to this, $\hat{\theta}_1(s)$ and $\hat{\theta}_2(s)$ can be factorised out, and with the use of partial fraction decomposition, simplified into a form where inverse transform pair can be observed [3]

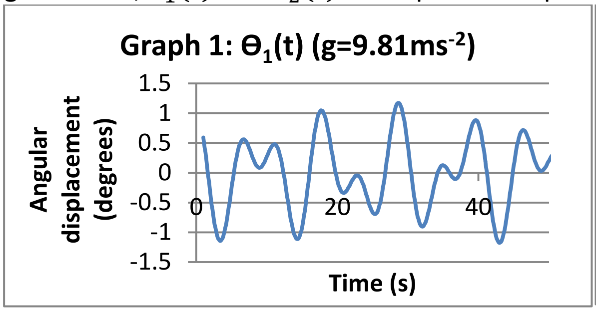
Results

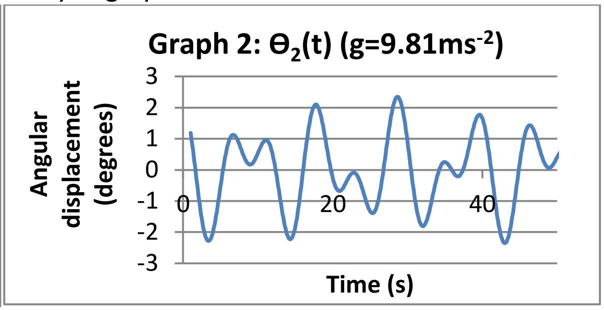
Having extraneously obtained the form for equations 2 and 3 exposing the transform pairs, a table of standard Laplacian pairs can be consulted to return to the time domain. The new form is now shown in equations 5 and 6 which fully describe the motion of the double pendulum in the limit of the small angle approximation:

$$\theta_1(t) = \tfrac{1}{2}Cos\left(\sqrt{\tfrac{g}{24}}t\right) + \tfrac{1}{2}Cos\left(\sqrt{\tfrac{g}{8}}t\right) + \tfrac{1}{\sqrt{2g}}Sin\left(\sqrt{\tfrac{g}{8}}t\right) - \tfrac{3}{\sqrt{6g}}Sin\left(\sqrt{\tfrac{g}{24}}t\right) \quad (5)$$

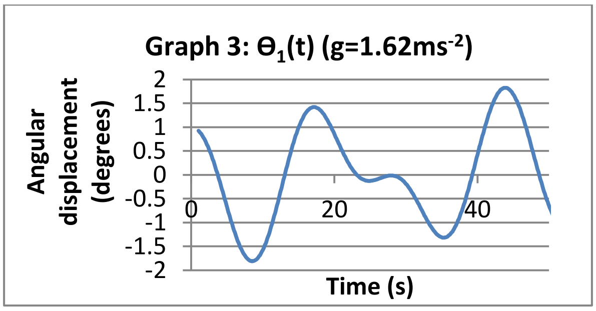
$$\theta_2(t) = Cos\left(\sqrt{\tfrac{g}{24}}t\right) + Cos\left(\sqrt{\tfrac{g}{8}}t\right) + \tfrac{2}{\sqrt{2g}}Sin\left(\sqrt{\tfrac{g}{8}}t\right) - \tfrac{6}{\sqrt{6g}}Sin\left(\sqrt{\tfrac{g}{24}}t\right). \quad (6)$$

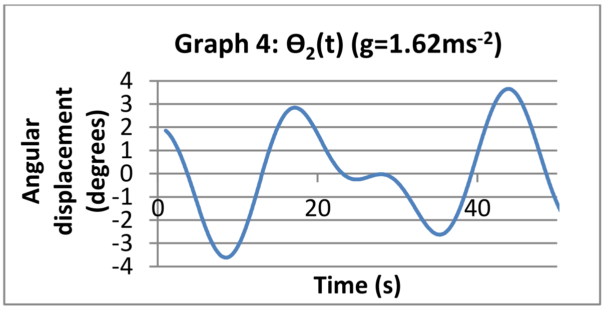
Interestingly, their similar form shows that the behaviour of the two pendulums stems from the same set of constituent waves, and differs only in their coefficients. For the standard value of g=9.81ms^{-2}, $\theta_1(t)$ and $\theta_2(t)$ are represented pictorially in graphs 1 and 2:





These data show the expected chaotic motion where the second pendulum is a small number of seconds out of phase from the first, and reaches amplitudes of approximately twice that of the first pendulum. The coupling has the effect of accentuating the motion of the second pendulum, making any subsequent trace on a seismograph much larger. When the gravity is set to the lunar value of 1.62ms^{-2}, we see from graphs 3 and 4 that the angular displacement reaches amplitudes almost twice that seen on earth:





This supports the idea that the increased ‘sensitivity’ of the pendulum on the moon could be viably used to detect moonquakes and infer details about lunar structure. This would replace the currently used geophones which are lacking in their maximum resolution and are dependent on circuitry for continued operation. As a result of this decrease in the complexity and other limiting factors in currently employed lunar instruments, commensurately more detailed and reliable information could be gathered. As an extension to this, the effect of increasing amplitude is inversely proportional to gravity such that this method could provide yet more detailed information about the interior of low gravity bodies such as comets and meteors, which combined with shock generating devices used to induce wave propagation throughout the subject body.

References

[1]http://upload.wikimedia.org/wikipedia/commons/7/78/Double-Pendulum.svg - 16/10/14

[2]http://www.saylor.org/site/wp-content/uploads/2013/04/ME401-1.2.2-LaplaceTransform.pdf - 16/10/14

[3]http://eecs.ucf.edu/~abehal/eel5669/handouts/laplace.pdf - 16/10/14

[4]http://www.capital.edu/uploadedFiles/Capital/Academics/Schools_and_Departments/Natural_Sciences,_Nursing_and_Health/Computational_Studies/Educational_Materials/Mathematics/Solving%20Differential%20Equations%20using%20the%20Laplace%20Transformation.pdf – 16/10/14