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P4_6 Orbiting a Black Hole

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Abstract
During the BBC's classic science fiction show Doctor Who, the Doctor and his assistant Rose land the TARDIS on a planet that is orbiting a black hole – an 'impossible situation' according to the Doctor. This paper calculates the minimum radius and required angular momentum of the closest stable circular orbit around a Schwarzschild black hole of comparable mass to the one at the centre of the Milky Way (named Sagittarius A*). The minimum radius is found to be 3.54×10^{10}m while the angular momentum required to sustain this orbit is $2.04 \times 10^{10} m^2 s^{-1} kg^{-1}$.

Introduction
During the Doctor Who episode entitled 'The Impossible Planet' the Doctor and Rose land on a planet orbiting a black hole. This paper sets out to derive the closest stable circular orbit of a black hole with a similar mass to the one at the centre of our galaxy (Sagittarius A*) and the angular momentum the 'Impossible Planet' would require to sustain this orbit. The paper assumes the black hole is spherically symmetric, uncharged, non-rotating and in empty space [1]. Under these conditions, the curvature of space-time due to the black hole can be described by the Schwarzschild metric[1]:

$$ds^2 = (1 - \frac{2GM}{rc^2})c^2 dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta \, d\varphi^2). \quad (1)$$

Here r is the distance from the origin (the centre of the black hole), θ is the inclination from the vertical axis, φ is the azimuthal angle 'around' the axis, t is time, G is the gravitational constant - $6.67 \times 10^{-11} m^3 kg^{-1} s^{-2}$, M is the mass of the black hole and s is the space-time interval. This paper first finds the Lagrangian of the Schwarzschild metric in order to derive an equation for the radial motion of an orbiting object. From this an equation for the angular momentum of an orbiting body is found. Finally the minimum radius is calculated by considering the energy of the system.

Calculation of the minimum radius and required angular momentum
Assuming the orbit is in a plane and setting θ equal to 90°, the $d\theta^2$ term in (1) is equal to 0 and the $\sin^2\theta$ term is equal to 1. Furthermore, the space-time interval s is equivalent to a minimum length scale or alternatively a geodesic [2]; the path followed by a circular orbit in a spherically symmetric space-time. The four-velocity (the tangent vector of the worldline, or path, of an object in space-time [1]) of this circular orbit is therefore equal to s differentiated by the time elapsed along the geodesic followed by a massive particle (a time-like geodesic). This is the proper time τ [1]. The kinetic energy per unit mass is therefore equal to the four-velocity squared multiplied by ½.

Due to the fact there is no 'potential energy' term in the metric [2], the Lagrangian is given solely by the 'kinetic energy' term and is given below[2]:

$$L = \frac{1}{2}\left(\frac{ds}{d\tau}\right)^2 = \left(\frac{1}{2} - \frac{m}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \frac{r^2}{2}\left(\frac{d\varphi}{d\tau}\right)^2, \quad (2)$$

where m = GM/c^2. In order to find an expression for the radial motion of a particle orbiting in Schwarzschild space-time, the Euler-Lagrange differential equation must be evaluated for both φ and t. The Euler-Lagrange differential equation is given by[3]:

$$\frac{\partial L}{\partial x^i} - \frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^i} = 0, \quad (i = 0,1,2,3), \qquad (3)$$

where $\dot{x^i} = dx^i/d\tau$ and for the Schwarzschild metric; $x^0 = t,\ x^1 = r,\ x^2 = \theta$ and $x^3 = \varphi$. Evaluating (3) for φ and t gives the following expressions:

$$-\left(1 - \frac{2m}{r}\right)c^2 \frac{\partial^2 t}{\partial \tau^2} = 0 \ or \ \left(1 - \frac{2m}{r}\right)c^2 \frac{\partial t}{\partial \tau} = E, \qquad (4)$$

$$r^2 \frac{\partial^2 \varphi}{\partial \tau^2} = 0 \ or \ r^2 \frac{\partial \varphi}{\partial \tau} = A, \qquad (5)$$

where E and A are constants. It is apparent A is equal to angular momentum per unit mass. On the other hand E is the relativistic energy per unit mass of a particle [4]. Substituting the expressions for E and A into (2) and rearranging for $(\partial r/\partial \tau)^2$ yields:

$$\left(\frac{\partial r}{\partial \tau}\right)^2 = E^2 - c^2\left(1 + \frac{A^2}{r^2}\right)\left(1 - \frac{2m}{r}\right) = E^2 - {V_{eff}}^2, \qquad (6)$$

where V_{eff} is equal to the effective potential [4] of the orbit. For a circular orbit $(\partial r/\partial \tau)^2$ = 0 due to the fact r(τ) = constant. This means E = V_{eff} and the energy lies along an effective potential curve [4]. Assuming there are no external forces, the total energy E is required to be constant during an orbit and therefore it should not change with r. Hence $\partial V_{eff}/\partial r$ is equal to 0 which results in the expression:

$$\frac{2A^2(r - 3m) - 2mr^2}{r^4} = 0. \qquad (7)$$

Using (7), both r and A can be derived. r is found using the quadratic formula:

$$r = \frac{A^2}{2m} \pm \frac{1}{2}\sqrt{-12A^2 + \frac{A^4}{m^2}} \ . \qquad (8)$$

For $A^2 < 12m^2$, the answer is complex hence there is no solution. Above this value, there are two solutions corresponding to the $\pm$ sign, which in turn correspond to the maxima and minima of V_{eff} [4]. Minima correspond to stable orbits, while maxima correspond to unstable orbits. When $A^2 = 12m^2$, this represents a point of inflection and is the closest stable circular orbit to the black hole [4]. However, due to the fact the orbit lies on an inflection point, a small alteration towards the black hole would cause the object to fall inwards. Using (8), when $A^2 = 12m^2$; $r = 6m$.

Assuming Sagittarius A* has a mass of 4 million solar masses [5], this corresponds to a radius of $3.54\text{x}10^{10}$m. The angular momentum per unit mass required to maintain this orbital radius around Sagittarius A* is given by $\sqrt{12m^2}$ and is equal to 2.04 $\text{x}10^{10}\text{m}^2\text{s}^{-1}\text{kg}^{-1}$.

Conclusion

To conclude, provided 'The Impossible Planet' had a minimum angular momentum per unit mass of 2.04 $\text{x}10^{10}\text{m}^2\text{s}^{-1}\text{kg}^{-1}$, it could orbit as close as $3.54\text{x}10^{10}$m to a Schwarzschild black hole with the same mass as Sagittarius A*, without falling inwards. However this orbit is only marginally stable and the tiniest alteration could cause the planet to fall towards the event horizon. Further research could be undertaken into the energy required to sustain such an orbit.

References

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